A SHORT INTRODUCTION 
TO BASIC ASPECTS 
OF CONTINUUM MICROMECHANICS

Helmut J. Böhm
Institute of Lightweight Design and Structural Biomechanics (ILSB) 
Vienna University of Technology

updated: January 9, 2020
© Helmut J. Böhm, 1998, 2020
Contents

Notes on this Document iv

1 Introduction 1
  1.1 Inhomogeneous Materials 1
  1.2 Homogenization and Localization 4
  1.3 Volume Elements 6
  1.4 Overall Behavior, Material Symmetries 7
  1.5 Major Modeling Strategies in Continuum Micromechanics of Materials 9
  1.6 Model Verification and Validation 13

2 Mean-Field Methods 16
  2.1 General Relations between Mean Fields in Thermoelastic Two-Phase Materials 16
  2.2 Eshelby Tensor and Dilute Matrix–Inclusion Composites 20
  2.3 Some Mean-Field Methods for Thermoelastic Composites with Aligned Reinforcements 23
    2.3.1 Effective Field Approaches 24
    2.3.2 Effective Medium Approaches 30
  2.4 Other Analytical Estimates for Elastic Composites with Aligned Reinforcements 32
  2.5 Mean-Field Methods for Nonlinear and Inelastic Composites 35
    2.5.1 Viscoelastic Composites 35
    2.5.2 (Thermo-)Elastoplastic Composites 36
  2.6 Mean-Field Methods for Composites with Nonaligned Reinforcements 41
  2.7 Mean-Field Methods for Non-Ellipsoidal Reinforcements 45
  2.8 Mean-Field Methods for Diffusion-Type Problems 48

3 Bounding Methods 54
  3.1 Classical Bounds 54
  3.2 Improved Bounds 58
  3.3 Bounds on Nonlinear Mechanical Behavior 59
  3.4 Bounds on Diffusion Properties 60
  3.5 Comparisons of Mean-Field and Bounding Predictions for Effective Thermoelastic Moduli 60
  3.6 Comparisons of Mean-Field and Bounding Predictions for Effective Conductivities 67
4 General Remarks on Modeling Approaches Based on Discrete Microstructures 69
4.1 Microgeometries and Volume Elements ............................... 69
4.2 Boundary Conditions ................................................... 76
4.3 Numerical Engineering Methods ...................................... 78
4.4 Evaluation of Results .................................................... 84

5 Periodic Microfield Models ........................................... 88
5.1 Basic Concepts of Periodic Models ................................. 88
5.2 Boundary Conditions ................................................... 90
5.3 Application of Loads and Evaluation of Fields .................... 98
5.4 Periodic Models for Composites Reinforced by Continuous Fibers 102
5.5 Periodic Models for Composites Reinforced by Short Fibers .......... 108
5.6 Periodic Models for Particle Reinforced Composites ............... 114
5.7 Periodic Models for Porous and Cellular Materials ............... 120
5.8 Periodic Models for Some Other Inhomogeneous Materials .......... 125
5.9 Periodic Models Models for Diffusion-Type Problems ............. 127

6 Windowing Approaches ............................................... 129

7 Embedding Approaches and Related Models .......................... 134

8 Hierarchical and Multi-Scale Models .................................. 140

9 Closing Remarks .......................................................... 144

Bibliography ..................................................................... 146
Notes on this Document

The present document is an expanded and modified version of the CDL–FMD report 3–1998, “A Short Introduction to Basic Aspects of Continuum Micromechanics”, which, in turn, is based on lecture notes prepared for the European Advanced Summer Schools Frontiers of Computational Micromechanics in Industrial and Engineering Materials held in Galway, Ireland, in July 1998 and in August 2000. Related lecture notes were used for graduate courses in a Summer School held at Ameland, the Netherlands, in October 2000 and during the COMMAS Summer School held in September 2002 at Stuttgart, Germany. All of the above documents, in turn, are closely related to the micromechanics section of the lecture notes for the course “Composite Engineering” (317.003) offered regularly at Vienna University of Technology.

The course notes “A Short Introduction to Continuum Micromechanics” (Böhm, 2004) for the CISM Course on Mechanics of Microstructured Materials held in July 2003 in Udine, Italy, may be regarded as a compact version of the present report that employs a somewhat different notation. The course notes “Analytical and Numerical Methods for Modeling the Thermomechanical and Thermophysical Behavior of Microstructured Materials” (Böhm et al., 2009) are also related to the present report, but emphasize different aspects of continuum micromechanics, among them the thermal conduction behavior of inhomogeneous materials and the modeling of cellular materials.

The present document is being updated continuously to reflect current developments in continuum micromechanics as seen by the author. The latest version can be downloaded as https://www.ilsb.tuwien.ac.at/links/downloads/ilsbrep206.pdf. The August 12, 2015 version of this report can be accessed via the DOI: 10.13140/RG.2.1.3025.7127.

Voigt/Nye notation is used for the mechanical variables in chapters 1 to 3. Tensors of order 4, such as elasticity, compliance, concentration and Eshelby tensors, are written as 6×6 quasi-matrices, and stress- as well as strain-like tensors of order 2 as 6-(quasi-)vectors. These 6-vectors are connected to index notation by the relations

\[
\sigma = \begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{pmatrix} \leftrightarrow \begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{pmatrix}, \quad \varepsilon = \begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{pmatrix} \leftrightarrow \begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{13} \\
\gamma_{23}
\end{pmatrix},
\]

iv
where \( \gamma_{ij} = 2\varepsilon_{ij} \) are the shear angles\(^1\) ("engineering shear strains"). Tensors of order 4 are denoted by bold upper case letters, stress- and strain-like tensors of order 2 by bold lower case Greek letters, and 3-vectors by bold lower case letters. Conductivity-like tensors of order 2 are treated as 3×3 matrices and denoted by calligraphic upper case letters. All other variables are taken to be scalars.

In using the present notation it is assumed that the 4th order tensors show orthotropic or higher symmetry and that the material axes are aligned with the coordinate system where appropriate. Details of the ordering of the components of \( \varepsilon \) and \( \sigma \) will not impact formulae. It is worth noting that with a notation like the present one some coefficients of Eshelby tensors differ compared to index notation, compare Pedersen (1983).

The tensorial product between two tensors of order 2 as well as the dyadic product between two vectors are denoted by the symbol "\( \otimes \)"; where \([\eta \otimes \zeta]_{ijkl} = \eta_{ij}\zeta_{kl}\), and \([a \otimes b]_{ij} = a_i b_j\), respectively. The contraction between a tensor of order 2 and a 3-vector is denoted by the symbol "\( * \)"; where \([\zeta * n]_i = \zeta_{ij} n_j\). Other products are defined implicitly by the types of quasi-matrices and quasi-vectors involved. A superscript \( ^T \) denotes the transpose of a tensor or vector.

Constituents (phases) are denoted by superscripts, with \( ^{gp} \) standing for a general phase, \( ^{m} \) for a matrix, \( ^{i} \) for inhomogeneities of general shape, and \( ^{f} \) for fibers. Axial and transverse properties of transversely isotropic materials are marked by subscripts \( _A \) and \( _T \), respectively, and effective (or apparent) properties are denoted by a superscript asterisk \( ^* \).

The present use of phase volume fractions as microstructural parameters is valid for the microgeometries typically found in composite, porous and polycrystalline materials. Note that for penny shaped cracks a crack density parameter (O’Connell and Budiansky, 1974) is the proper choice.

\(^1\)This notation allows strain energy densities to be obtained as the scalar product \( U = \frac{1}{2} \sigma^T \varepsilon \). Furthermore, the stress and strain “rotation tensors” \( R_\sigma \) and \( R_\varepsilon \) follow the relationship \( R_\varepsilon = [(R_\sigma)^{-1}]^T \).
Chapter 1

Introduction

In the present report some basic issues of and some of the modeling strategies used for studying static and quasistatic problems in continuum micromechanics of materials are discussed. The main emphasis is put on application related (or “engineering”) aspects, and neither a comprehensive theoretical treatment nor a review of the pertinent literature are attempted. For more formal treatments of many of the concepts and methods discussed in the present work see, e.g., Mura (1987), Aboudi (1991), Nemat-Nasser and Hori (1993), Suquet (1997), Markov (2000), Bornert et al. (2001), Torquato (2002), Milton (2002), Qu and Cherkaoui (2006), Buryachenko (2007), Kachanov and Sevostianov (2013) as well as Kachanov and Sevostianov (2018); Yvonnet (2019) is specialized to numerical continuum micromechanics. Shorter overviews of continuum micromechanics were given, e.g., by Hashin (1983), Zaoui (2002) and Kanouté et al. (2009). Discussions of the history of the development of the field can be found in Markov (2000) and Zaoui (2002).

Due to the author’s research interests, more room is given to the thermomechanical behavior of two-phase materials showing a matrix–inclusion topology (“solid dispersions”) and, especially, to metal matrix composites (MMCs) than to materials with other phase topologies or phase geometries, or to multi-phase materials.

1.1 Inhomogeneous Materials

Many industrial and engineering materials as well as the majority of biological materials are inhomogeneous, i.e., they consist of dissimilar constituents (“phases”) that are distinguishable at some (small) length scale(s). Each constituent shows different material properties and/or material orientations and may itself be inhomogeneous at some smaller length scale(s). Inhomogeneous materials (also referred to as microstructured, heterogeneous or complex materials) play important roles in materials science and technology. Well-known examples of such materials are composites, concrete, polycrystalline materials, porous and cellular materials, functionally graded materials, wood, and bone.

The behavior of inhomogeneous materials is determined, on the one hand, by the relevant material properties of the constituents and, on the other hand, by their geometry and topology (the “phase arrangement”). Obviously, the availability of information on
these counts determines the accuracy of any model or theoretical description. The behavior of inhomogeneous materials can be studied at a number of length scales ranging from sub-atomic scales, which are dominated by quantum effects, to scales for which continuum descriptions are best suited. The present report concentrates on continuum models for heterogeneous materials, the pertinent research field being customarily referred to as continuum micromechanics of materials. The use of continuum models puts a lower limit on the length scales that can be covered with the methods discussed here, which typically may be taken as being of the order of 1 µm (Forest et al., 2002).

As will be discussed in section 1.2, an important aim of bridging length scales for inhomogeneous materials lies in deducing their overall (“effective” or “apparent”) behavior (e.g., stiffness, thermal expansion and strength properties, heat conduction and related transport properties, electrical and magnetic properties, electromechanical properties, etc.) from the corresponding material behavior of the constituents (as well as that of the interfaces between them) and from the geometrical arrangement of the phases. Such scale transitions from lower (finer) to higher (coarser) length scales aim at achieving a marked reduction in the number of degrees of freedom describing the system. The continuum methods discussed in this report are most suitable for handling scale transitions from length scales in the low micrometer range to macroscopic samples, components or structures with sizes of millimeters to meters.

In what follows, the main focus will be put on describing the thermomechanical behavior of inhomogeneous two-phase materials. Most of the discussed modeling approaches can, however, be extended to multi-phase materials in a straightforward way, and there is a large body of literature applying analogous or related continuum methods to other physical properties of inhomogeneous materials, compare, e.g., Hashin (1983), Torquato (2002) as well as Milton (2002), and see sections 2.8 and 5.9 of the present report.

The most basic classification criterion for inhomogeneous materials is aimed at the microscopic phase topology. In matrix–inclusion arrangements (“solid suspensions”, as found in particulate and fibrous materials, such as “typical” composite materials, in porous materials, and in closed-cell foams) only the matrix shows a connected topology and the constituents play clearly distinct roles. In interpenetrating (interwoven, co-continuous, skeletal) phase arrangements (as found, e.g., in open-cell foams, in certain composites, or in many functionally graded materials) and in typical polycrystals (“granular materials”), in contrast, the phases cannot be readily distinguished topologically.

Obviously, an important parameter in continuum micromechanics is the level of inhomogeneity of the constituents’ behavior, which is often described by a phase contrast. For example, the elastic contrast of a two-phase composite is typically defined in terms of the Young’s moduli of matrix and inhomogeneities (or reinforcements), $E^{(i)}$ and $E^{(m)}$, as

---

2The designation “effective material properties” is typically reserved for describing the macroscopic responses of bulk materials, whereas the term “apparent material properties” is used for the properties of finite-sized samples (Huet, 1990), for which boundary effects play a role.

3With respect to their thermomechanical behavior, porous and cellular materials can usually be treated as inhomogeneous materials in which one constituent shows vanishing stiffness, thermal expansion, conductivity, etc., compare section 5.7.
\[ c_{el} = \frac{E^{(i)}}{E^{(m)}} \].

### Length Scales

In the present context the lowest length scale described by a given micromechanical model is termed the microscale, the largest one the macroscale and intermediate ones are called mesoscales. The fields describing the behavior of an inhomogeneous material, i.e., in mechanics the stresses \( \sigma(x) \), strains \( \varepsilon(x) \) and displacements \( u(x) \), are split into contributions corresponding to the different length scales, which are referred to as micro-, macro- and mesofields, respectively. The phase geometries on the meso- and microscales are denoted as meso- and microgeometries.

Most micromechanical models are based on the assumption that the length scales in a given material are well separated. This is understood to imply that for each micro–macro pair of scales, on the one hand, the fluctuating contributions to the fields at the smaller length scale ("fast variables") influence the behavior at the larger length scale only via their volume averages. On the other hand, gradients of the fields as well as compositional gradients at the larger length scale ("slow variables") are not significant at the smaller length scale, where these fields appear to be locally constant and can be described in terms of uniform “applied fields” or “far fields”. Formally, this splitting of the strain and stress fields into slow and fast contributions can be written as

\[ \varepsilon(x) = \langle \varepsilon \rangle + \varepsilon'(x) \quad \text{and} \quad \sigma(x) = \langle \sigma \rangle + \sigma'(x) \],

where \( \langle \varepsilon \rangle \) and \( \langle \sigma \rangle \) are the macroscopic, averaged fields, whereas \( \varepsilon'(x) \) and \( \sigma'(x) \) stand for the microscopic fluctuations.

Unless specifically stated otherwise, in the present report the above conditions on the slow and fast variables are assumed to be met. If this is not the case to a sufficient degree (e.g., in the cases of insufficiently separated length scales, of the presence of marked compositional or load gradients, of regions in the vicinity of free surfaces of inhomogeneous materials, or of macroscopic interfaces adjoined by at least one inhomogeneous material), embedding schemes, compare chapter 7, or special analysis methods must be applied. The latter may take the form of second-order homogenization schemes that explicitly account for deformation gradients on the microscale (Kouznetsova et al., 2004) and result in non-local homogenized behavior, see, e.g., Feyel (2003).

Smaller length scales than the ones considered in a given model may or may not be amenable to continuum mechanical descriptions. For an overview of methods applicable below the continuum range see, e.g., Raabe (1998).

---

4The nomenclature with respect to “micro”, “meso” and “macro” is far from universal, the naming of the length scales of inhomogeneous materials being notoriously inconsistent even in literature dealing specifically with continuum micromechanics. In a more neutral way, the smaller length scale may be referred to as the “fine-grained” and the larger one as the “coarse-grained” length scale.
1.2 Homogenization and Localization

The two central aims of continuum micromechanics can be stated to be, on the one hand, the bridging of length scales and, on the other hand, studying the structure–property relationships of inhomogeneous materials.

The first of the above issues, the bridging of length scales, involves two main tasks. On the one hand, the behavior at some larger length scale (the macroscale) must be estimated or bounded by using information from a smaller length scale (the microscale), i.e., homogenization problems must be solved. The most important applications of homogenization are materials characterization, i.e., simulating the overall material response under simple loading conditions such as uniaxial tensile tests, and constitutive modeling, where the responses to general loads, load paths and loading sequences must be described. Homogenization (also referred to as “upscale” or “coarse graining”) may be interpreted as describing the behavior of a material that is inhomogeneous at some lower length scale in terms of a (fictitious) energetically equivalent, homogeneous reference material at some higher length scale, which is sometimes referred to as the homogeneous equivalent medium (HEM). Since homogenization links the phase arrangement at the microscale to the macroscopic behavior, it will, in general, provide microstructure–property relationships. On the other hand, the local responses at the smaller length scale may be deduced from the loading conditions (and, where appropriate, from the load histories) on the larger length scale. This task, which corresponds to “zooming in” on the local fields in an inhomogeneous material, is referred to as localization, downscaling or fine graining. In either case the main inputs are the geometrical arrangement and the material behaviors of the constituents at the microscale. In many continuum micromechanical methods, homogenization is less demanding than localization because the local fields tend to show a much more marked dependence on details of the local geometry of the constituents.

For a volume element $\Omega_s$ of an inhomogeneous material that is sufficiently large, contains no significant gradients of composition, and shows no significant variations in the applied loads, homogenization relations take the form of volume averages of some variable $f(x)$,

$$\langle f \rangle = \frac{1}{\Omega_s} \int_{\Omega_s} f(x) \, d\Omega .$$

(1.3)

The homogenization relations for the stress and strain tensors can be given as

$$\langle \varepsilon \rangle = \frac{1}{\Omega_s} \int_{\Omega_s} \varepsilon(x) \, d\Omega = \frac{1}{2\Omega_s} \int_{\Gamma_s} \left[ u(x) \otimes n(x) + n(x) \otimes u(x) \right] \, d\Gamma$$

$$\langle \sigma \rangle = \frac{1}{\Omega_s} \int_{\Omega_s} \sigma(x) \, d\Omega = \frac{1}{\Omega_s} \int_{\Gamma_s} t(x) \otimes x \, d\Gamma ,$$

(1.4)

where $\Gamma_s$ stands for surface of the volume element $\Omega_s$, $u(x)$ is the displacement vector, $t(x) = \sigma(x) * n(x)$ is the surface traction vector, and $n(x)$ is the surface normal vector. Equations (1.4) are known as the average strain and average stress theorems, and the surface integral formulation for $\varepsilon$ given above pertains to the small strain regime and to continuous displacements. Under the latter condition the mean strains and stresses in a control volume, $\langle \varepsilon \rangle$ and $\langle \sigma \rangle$, are fully determined by the surface displacements and tractions. If the displacements show discontinuities, e.g., due to imperfect interfaces between
the constituents or due to (micro) cracks, correction terms involving the displacement jumps across imperfect interfaces or cracks must be introduced, compare Nemat-Nasser and Hori (1993). In the absence of body forces the microstresses \( \sigma(x) \) are self-equilibrated (but not necessarily zero). In the above form, eqn. (1.4) applies to linear elastic behavior, but it can be modified to cover the thermoelastic regime and extended into the nonlinear range, e.g., to elastoplastic materials described by secant or incremental plasticity models, compare section 2.5. For a discussion of averaging techniques and related results for finite deformation regimes see, e.g., Nemat-Nasser (1999).

The microscopic strain and stress fields, \( \varepsilon(x) \) and \( \sigma(x) \), in a given volume element \( \Omega_s \) are formally linked to the corresponding macroscopic responses, \( \langle \varepsilon \rangle \) and \( \langle \sigma \rangle \), by localization (or projection) relations of the type

\[
\varepsilon(x) = A(x) \langle \varepsilon \rangle \quad \text{and} \quad \sigma(x) = B(x) \langle \sigma \rangle.
\]  

(1.5)

\( A(x) \) and \( B(x) \) are known as mechanical strain and stress concentration tensors (or influence functions Hill (1963), influence tensors), respectively. When these are known, localization tasks can obviously be carried out.

Equations (1.2) and (1.4) imply that the volume averages of fluctuations vanish for sufficiently large integration volumes,

\[
\langle \varepsilon' \rangle = \frac{1}{\Omega_s} \int_{\Omega_s} \varepsilon'(x) \, d\Omega = 0 \quad \text{and} \quad \langle \sigma' \rangle = \frac{1}{\Omega_s} \int_{\Omega_s} \sigma'(x) \, d\Omega = 0
\]  

(1.6)

Similarly, surface integrals over the microscopic fluctuations of the field variables tend to zero for appropriate volume elements\(^5\).

For suitable volume elements of inhomogeneous materials that show sufficient separation between the length scales and for suitable boundary conditions the relation

\[
\frac{1}{2} \langle \sigma^T \varepsilon \rangle = \frac{1}{2\Omega} \int_{\Omega} \tilde{\sigma}^T(x) \tilde{\varepsilon}(x) \, d\Omega = \frac{1}{2} \langle \tilde{\sigma} \rangle^T \langle \tilde{\varepsilon} \rangle
\]  

(1.7)

must hold for general statically admissible stress fields \( \tilde{\sigma} \) and kinematically admissible strain fields \( \tilde{\varepsilon} \), compare Hill (1967). This equation is known as Hill’s macrohomogeneity condition, the Hill–Mandel condition or the energy equivalence condition, compare Nemat-Nasser (1999), Bornert (2001) and Zaoui (2001). Equation (1.7) states that the strain energy density of the microfields equals the strain energy density of the macrofields, making the microscopic and macroscopic descriptions energetically equivalent. In other words, the fluctuations of the microfields do not contribute to the macroscopic strain energy, i.e.,

\[
\langle \sigma^T \varepsilon' \rangle = 0
\]  

(1.8)

The Hill–Mandel condition forms the basis of the interpretation of homogenization procedures in the thermoelastic regime\(^6\) in terms of a homogeneous comparison material (or “reference medium”) that is energetically equivalent to a given inhomogeneous material.

\(^5\)Whereas volume and surface integrals over products of slow and fast variables vanish, integrals over products of fluctuating variables (“correlations”), e.g., \( \langle \varepsilon'^2 \rangle \), do not vanish in general.

\(^6\)For a discussion of the Hill–Mandel condition for finite deformations and related issues see, e.g., Khisaeva and Ostoja-Starzewski (2006).
1.3 Volume Elements

The second central issue of continuum micromechanics, studying the structure–property relationships of inhomogeneous materials, obviously requires suitable descriptions of their structure at the appropriate length scale, i.e., within the present context, their microgeometries.

The microgeometries of real inhomogeneous materials are at least to some extent random and, in the majority of cases of practical relevance, their detailed phase arrangements are highly complex. As a consequence, exact expressions for \( A(x), B(x), \varepsilon(x), \sigma(x), \) etc., in general cannot be given with reasonable effort, and approximations have to be introduced. Typically, these approximations are based on the ergodic hypothesis, i.e., the heterogeneous material is assumed to be statistically homogeneous. This implies that sufficiently large volume elements selected at random positions within the sample have statistically equivalent phase arrangements and give rise to the same averaged material properties\(^7\). As mentioned above, such material properties are referred to as the overall or effective material properties of the inhomogeneous material.

Ideally, the homogenization volume should be chosen to be a proper representative volume element (RVE), i.e., a subvolume of \( \Omega_s \) that is of sufficient size to contain all information necessary for describing the behavior of the composite. Representative volume elements can be defined, on the one hand, by requiring them to be statistically representative of the microgeometry, the resulting “geometrical RVEs” being independent of the physical property to be studied. On the other hand, the definition can be based on the requirement that the overall responses with respect to some given physical behavior do not depend on the actual position of the RVE nor on the boundary conditions applied to it (Hill, 1963). The size of such “physical RVEs” depends on both the physical property considered and on the microgeometry, the Hill–Mandel condition, eqn. (1.7), having to be fulfilled by definition.

An RVE must be sufficiently large to allow a meaningful sampling of the microfields and sufficiently small for the influence of macroscopic gradients to be negligible. In addition, it must be smaller than a typical sample or components\(^8\). For further discussion of microgeometries and homogenization volumes see section 4.1. For methods involving the analysis of discrete volume elements, the latter must be sufficiently small for an analysis of the microfields to be feasible.

The fields in a given constituent\(^9\) can be split into phase averages and fluctuations by analogy to eqn. (1.2) as

---

\(^7\)Some inhomogeneous materials are not statistically homogeneous by design, e.g., functionally graded materials in the direction(s) of the gradient(s), and, consequently, may require nonstandard treatment. For such materials it is not possible to define effective material properties in the sense of eqn. (1.10). Deviations from statistical homogeneity may also be introduced into inhomogeneous materials as side effects of manufacturing processes.

\(^8\)This requirement was symbolically denoted as MICRO\( \ll \)MESO\( \ll \)MACRO by Hashin (1983), where MICRO and MACRO have their “usual” meanings and MESO stands for the length scale of the homogenization volume. As noted by Nemat-Nasser (1999) it is the dimension relative to the microstructure relevant for a given problem that is important for the size of an RVE.

---
\[
\varepsilon^{(p)}(x) = \langle \varepsilon \rangle^{(p)} + \varepsilon^{(p)\prime}(x) \quad \text{and} \quad \sigma^{(p)}(x) = \langle \sigma \rangle^{(p)} + \sigma^{(p)\prime}(x) .
\]

In the case of materials with matrix–inclusion topology, i.e., for typical composites, analogous relations can be also be defined at the level of individual inhomogeneities. The variations of the (average) fields between individual particles or fibers are known as inter-particle or inter-fiber fluctuations, respectively, whereas field gradients within given inhomogeneities give rise to intra-particle and intra-fiber fluctuations.

Needless to say, volume elements should be chosen to be as simple possible in order to limit the modeling effort, but their level of complexity must be sufficient for covering the aspects of their behavior targeted by a given study, compare, e.g., the examples given by Forest et al. (2002).

### 1.4 Overall Behavior, Material Symmetries

The homogenized strain and stress fields of an elastic inhomogeneous material as obtained by eqn. (1.4), \( \langle \varepsilon \rangle \) and \( \langle \sigma \rangle \), can be linked by effective elastic tensors \( E^* \) and \( C^* \) as

\[
\langle \sigma \rangle = E^* \langle \varepsilon \rangle \quad \text{and} \quad \langle \varepsilon \rangle = C^* \langle \sigma \rangle ,
\]


which may be viewed as the elasticity and compliance tensors, respectively, of an appropriate equivalent homogeneous material, with \( C^* = E^*^{-1} \). Using eqns. (1.4) and (1.5) these effective elastic tensors can be obtained from the local elastic tensors, \( E(x) \) and \( C(x) \), and the concentration tensors, \( A(x) \) and \( B(x) \), by volume averaging

\[
E^* = \frac{1}{\Omega_s} \int_{\Omega_s} E(x)A(x)d\Omega
\]

\[
C^* = \frac{1}{\Omega_s} \int_{\Omega_s} C(x)B(x)d\Omega ,
\]

Other effective properties of inhomogeneous materials, e.g., tensors describing their thermophysical behavior, can be evaluated in an analogous way.

The resulting homogenized behavior of many multi-phase materials can be idealized as being statistically isotropic or quasi-isotropic (e.g., for composites reinforced with spherical particles, randomly oriented particles of general shape or randomly oriented fibers, many polycrystals, many porous and cellular materials, random mixtures of two phases) or statistically transversely isotropic (e.g., for composites reinforced with aligned fibers or platelets, composites reinforced with nonaligned reinforcements showing a planar random or other axisymmetric orientation distribution function, etc.), compare (Hashin, 1983). Of course, lower material symmetries of the homogenized response may also be found, e.g., in textured polycrystals or in composites containing reinforcements with orientation distributions of low symmetry, compare Allen and Lee (1990).

Statistically isotropic multi-phase materials show the same overall behavior in all directions, and their effective elasticity tensors and thermal expansion tensors take the forms
isotropic materials have the structure

\[ \xi = \begin{pmatrix} \xi_{11} & \xi_{12} & 0 & 0 \\ \xi_{12} & \xi_{11} & 0 & 0 \\ 0 & 0 & \xi_{44} & 0 \\ 0 & 0 & 0 & \xi_{44} \end{pmatrix} \]

and the fiber volume fraction behavior in the linear range (the effective coefficient of thermal expansion is isotropic parameters in this context are, e.g., the axial and transverse effective Young’s moduli, the effective shear modulus \( G^* = E_{44} = E^*/2(1 + \nu^*) \), the effective bulk modulus \( K^* = (E_{11}^* + 2E_{12}^*)/3 = (E^*/3)(1 - 2\nu^*) \), or the effective Lamé constants) and one is required for the effective thermal expansion behavior in the linear range (the effective coefficient of thermal expansion \( \alpha^* = \alpha_{11}^* \)).

The effective elasticity and thermal expansion tensors for statistically transversely isotropic materials have the structure

\[
\mathbf{E}^* = \begin{pmatrix} E_{11}^* & E_{12}^* & E_{12}^* & 0 & 0 & 0 \\ E_{12}^* & E_{22}^* & E_{23}^* & 0 & 0 & 0 \\ E_{12}^* & E_{23}^* & E_{22}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{44}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{46}^* \end{pmatrix} \quad \alpha^* = \begin{pmatrix} \alpha_A^* \\ \alpha_T^* \end{pmatrix},
\]

where 1 is the axial direction and 2–3 is the transverse plane of isotropy. Generally, the thermoelastic behavior of transversely isotropic materials is described by five independent elastic constants and two independent coefficients of thermal expansion. Appropriate elasticity parameters in this context are, e.g., the axial and transverse effective Young’s moduli, \( E_A^* = E_{11}^* - \frac{2E_{12}^*}{E_{22}^* + E_{23}^*} \) and \( E_T^* = E_{23}^* - \frac{E_{23}^* + E_{12}^*}{E_{11}^*} \), the axial and transverse effective shear moduli, \( G_A^* = E_{44}^* \) and \( G_T^* = E_{46}^* \), the axial and transverse effective Poisson’s ratios, \( \nu_A^* = \nu_{12} = \frac{E_{12}^*}{E_{22}^* + E_{23}^*} \) and \( \nu_T^* = \nu_{12} = \frac{E_{12}^*}{E_{11}^*} \), as well as the effective transverse (plane strain) bulk modulus \( K_T^* = (E_{22}^* + E_{23}^*)/2 = E_A^*/2(1 - \nu_T^*)/(E_A^*/E_T^*) - 2\nu_T^* \). The transverse (“in-plane”) properties are related via \( G_T^* = E_T^*/2(1 + \nu_T^*) \), but there is no general linkage between the axial properties \( E_A^*, G_A^* \) and \( \nu_A^* \) beyond the above definition of \( K_T^* \). For the special case of materials consisting of aligned phases that are continuous in the 1-direction, however, the Hill (1964) connections,

\[
E_A^* = \xi E_A^{(f)} + (1 - \xi)E_A^{(m)} + \frac{4(\nu_A^{(f)} - \nu_A^{(m)})^2}{(1/K_T^{(f)}) - 1/K_T^{(m)}} \left( \frac{\xi}{K_T^{(f)}} + \frac{1 - \xi}{K_T^{(m)}} - \frac{1}{K_T^*} \right)
\]

\[
\nu_A^* = \xi \nu_A^{(f)} + (1 - \xi)\nu_A^{(m)} + \frac{\nu_A^{(f)} - \nu_A^{(m)}}{1/K_T^{(f)} - 1/K_T^{(m)}} \left( \frac{\xi}{K_T^{(f)}} + \frac{1 - \xi}{K_T^{(m)}} - \frac{1}{K_T^*} \right)
\]

allow the effective moduli \( E_A^* \) and \( \nu_A^* \) to be expressed by \( K_T^* \), some constituent properties, and the fiber volume fraction \( \xi = \Omega^{(f)}/\Omega; \) see also Milton (2002). Analogous relations hold for unidirectionally reinforced composites of tetragonal macroscopic symmetry (Berggren...
et al., 2003). Both an axial and a transverse effective coefficient of thermal expansion, $\alpha_A^* = \alpha_{11}^*$ and $\alpha_T^* = \alpha_{22}^*$, are required for transversely isotropic materials.

The overall material symmetries of inhomogeneous materials and their effect on various physical properties can be treated in full analogy to the symmetries of crystals as discussed, e.g., by Nye (1957). Accordingly, deviations of predicted elastic tensors from macroscopically isotropic elastic symmetry can be assessed via a Zener (1948) anisotropy ratio, $Z = 2E_{66}^*/(E_{22}^* - E_{23}^*)$, or other anisotropy parameters, see, e.g., Kanit et al. (2006).

The influence of the overall symmetry of the phase arrangement on the overall mechanical behavior of inhomogeneous materials can be marked\(^9\), especially on the nonlinear responses to mechanical loads. Accordingly, it is good practice to aim at approximating the symmetry of the actual material as closely as possible in any modeling effort.

### 1.5 Major Modeling Strategies in Continuum Micromechanics of Materials

All micromechanical methods described in the present report can be used to carry out materials characterization, i.e., simulating the overall material response under simple loading conditions such as uniaxial tensile tests. Many homogenization procedures can also be employed directly to provide micromechanically based constitutive material models at higher length scales. This implies that they allow evaluating the full homogenized stress and strain tensors for any pertinent loading condition and for any pertinent loading history\(^10\). This task is obviously much more demanding than materials characterization. Compared to semi-empirical constitutive laws, as proposed, e.g., by Davis (1996), micromechanically based constitutive models have both a clear physical basis and an inherent capability for “zooming in” on the local phase stresses and strains by using localization procedures.

Evaluating the local responses of the constituents (in the ideal case, at any material point) for a given macroscopic state of a sample or structure, i.e., localization, is of special interest for identifying local deformation mechanisms and for studying as well as assessing local strength relevant behavior, such as the onset and progress of plastic yielding or of damage, which, of course, can have major repercussions on the macroscopic behavior. For valid descriptions of local strength-relevant responses details of the microgeometry tend to be of major importance and may, in fact, determine the macroscopic response, an extreme case being the mechanical strength of brittle inhomogeneous materials.

\(^9\)Overall properties described by tensors or lower rank, e.g., thermal expansion and thermal conduction, are less sensitive to material symmetry effects than are mechanical responses, compare Nye (1957).

\(^10\)The overall thermomechanical behavior of homogenized materials is often richer than that of the constituents, i.e., the effects of the interaction of the constituents in many cases cannot be satisfactorily described by simply adapting material parameters without changing the functional relationships in the constitutive laws of the constituents. For example, a composite consisting of a matrix that follows $J_2$ plasticity and elastic reinforcements shows some pressure dependence in its macroscopic plastic behavior, and the macroscopic flow behavior of inhomogeneous materials can lose normality even though that of each of the constituents is associated (Li and Ostoja-Starzewski, 2006). Also, two dissimilar constituents following Maxwell-type linear viscoelastic behavior in do not necessarily give rise to a macroscopic Maxwell behavior (Barello and Lévesque, 2008).
Because for realistic phase distributions an exact analysis of the spatial variations of the microfields in large volume elements tends to be beyond present capabilities\(^{11}\) suitable approximations must be introduced. For convenience, the majority of the resulting modeling approaches may be treated as falling into two groups. The first of these comprises methods that describe interactions, e.g., between phases or between individual reinforcements, in a collective way, its main representatives being

- **Mean-Field Approaches (MFAs)** and related methods (see chapter 2): Highly idealized microgeometries are used (compare, e.g., fig 2.1) and the microfields within each constituent are approximated by their phase averages \(\langle \varepsilon \rangle^{(p)}\) and \(\langle \sigma \rangle^{(p)}\), i.e., phase-wise uniform stress and strain fields are employed. The phase geometry enters these models, sometimes implicitly, via statistical descriptors, such as volume fractions, macroscopic symmetry, phase topology, reinforcement aspect ratios, etc. In MFAs the localization relations take the form

\[
\langle \varepsilon \rangle^{(p)} = \bar{A}^{(p)} \langle \varepsilon \rangle
\]
\[
\langle \sigma \rangle^{(p)} = \bar{B}^{(p)} \langle \sigma \rangle
\]

and the homogenization relations can be written as

\[
\langle \varepsilon \rangle^{(p)} = \frac{1}{\Omega^{(p)}} \int_{\Omega^{(p)}} \varepsilon(x) \, d\Omega \quad \text{with} \quad \langle \varepsilon \rangle = \sum_p V^{(p)} \langle \varepsilon \rangle^{(p)}
\]
\[
\langle \sigma \rangle^{(p)} = \frac{1}{\Omega^{(p)}} \int_{\Omega^{(p)}} \sigma(x) \, d\Omega \quad \text{with} \quad \langle \sigma \rangle = \sum_p V^{(p)} \langle \sigma \rangle^{(p)}
\]

where \( (p) \) denotes a given phase of the material, \( \Omega^{(p)} \) is the volume occupied by this phase, and \( V^{(p)} = \Omega^{(p)} / \sum_k \Omega^{(k)} = \Omega^{(p)} / \Omega_s \) is the volume fraction of the phase. In contrast to eqn. (1.5) the phase concentration tensors \( \bar{A} \) and \( \bar{B} \) used in MFAs are not functions of the spatial coordinates\(^{12}\).

Mean-field approaches tend to be formulated (and provide estimates for effective properties) in terms of the phase concentration tensors, they pose low computational requirements, and they have been highly successful in describing the thermoelastic response of inhomogeneous materials. Their use in modeling nonlinear composites continues to be a subject of active research. Their most important representatives are effective field and effective medium approximations.

- **Bounding Methods** (see chapter 3): Variational principles are used to obtain upper and (in many cases) lower bounds on the overall elastic tensors, elastic moduli, secant moduli, and other physical properties of inhomogeneous materials the microgeometries of which are described by statistical parameters. Many analytical bounds are obtained on the basis of phase-wise constant stress polarization fields, making them closely related to MFAs. Bounds — in addition to their intrinsic value — are tools of vital importance in assessing other models of inhomogeneous materials. Furthermore, in most cases one of the bounds provides useful estimates for the physical property under consideration, even if the bounds are rather slack (Torquato, 1991).

\(^{11}\) Exact predictions of the effective properties would require an infinite set of correlation functions for statistically characterizing the inhomogeneous microstructure, compare Torquato et al. (1999).

\(^{12}\) Surface integral formulations analogous to eqn. (1.4) may be used to evaluate consistent expressions for \( \langle \varepsilon \rangle^{(p)} \) for void-like and \( \langle \sigma \rangle^{(p)} \) for rigid inhomogeneities embedded in a matrix.
Because they do not explicitly account for pair-wise or multi-particle interactions mean-field approaches have sometimes been referred to as “non-interacting approximations” in the literature. This designator, however, is best limited to the dilute regime, collective interactions being explicitly incorporated into mean-field models for non-dilute volume fractions. MFA and bounding methods implicitly postulate the existence of a representative volume element.

The second group of approximations are based on studying discrete microgeometries, for which they aim at evaluating the microfields, thus fully accounting for the interactions between phases. It includes the following groups of models, compare the sketches in fig. 1.1.

- **Periodic Microfield Approaches (PMAs)**, often referred to as periodic homogenization schemes and sometimes as unit cell methods, see chapter 5. In such models the inhomogeneous material is approximated by an infinitely extended model material with a periodic phase arrangement. The resulting periodic microfields are usually evaluated by analyzing repeating unit cells (which may describe microgeometries ranging from rather simplistic to highly complex ones) via analytical or numerical methods. Such approaches are often used for performing materials characterization of inhomogeneous materials in the nonlinear range, but they can also be employed as micromechanically based constitutive models. The high resolution of the microfields provided by PMAs can be very useful in studying the initiation of damage at the microscale. However, because they inherently give rise to periodic configurations of damage and patterns of cracks, PMAs typically are not a good choice for investigating phenomena such as the interaction of the microgeometry with macroscopic cracks. Periodic microfield approaches can give detailed information on the local stress and strain fields within a given unit cell, but they tend to be computationally expensive. Among the methods in this group they are the only one that does not intrinsically give rise to boundary layers in the microfields.

- **Windowing Approaches** (see chapter 6): Subregions (“windows”) — usually, but not necessarily, of rectangular or hexahedral shape — are randomly chosen from a given phase arrangement and subjected to boundary conditions that guarantee energy equivalence between the micro- and macroscales. Accordingly, windowing methods describe the behavior of individual inhomogeneous samples rather than of inhomogeneous materials and give rise to apparent rather than effective macroscopic responses. For the special cases of macrohomogeneous stress and strain boundary conditions, respectively, lower and upper estimates for and bounds on the overall behavior of the inhomogeneous material can be obtained. In addition, mixed homogeneous boundary conditions can be applied in order to generate estimates.

- **Embedded Cell or Embedding Approaches (ECAs)**; see chapter 7: The inhomogeneous material is approximated by a model consisting of a “core” containing a discrete phase arrangement that is embedded within some outer region showing smeared-out material behavior; far field loads are applied to this outer region. The material properties of the embedding layer may be described by some macroscopic constitutive law, they can be determined self-consistently or quasi-self-consistently from the behavior of the core, or the embedding region may take the form of a coarse description and/or discretization of the phase arrangement. ECAs can be used for materials character-
ization, and they are usually the best choice for studying regions of special interest in inhomogeneous materials, such as the surroundings of tips of macroscopic cracks. Like PMAs, embedded cell approaches can resolve local stress and strain fields in the core region at high detail, but tend to be computationally expensive.

- Other homogenization approaches employing discrete microgeometries, such as the statistics-based non-periodic homogenization scheme of Li and Cui (2005).

Because the above group of methods explicitly study mesodomains as defined by Hashin (1983) they are sometimes referred to as “mesoscale approaches”, and because the microfields are evaluated at a high level of detail, the name “full field models” is often used. Figure 1.1 shows a sketch of a volume element as well as PMA, ECA and windowing approaches applied to it.

Some further descriptions that have been applied to studying the macroscopic thermo-mechanical behavior of inhomogeneous materials, such as isostain and isostress models
(the former being known as the “rules of mixture”) and the Halpin–Tsai equations are not discussed here because, with the exception of special cases, their connection to actual microgeometries is not very strong, limiting their predictive capabilities. For brevity, a number of micromechanical models with solid physical basis, such as expressions for self-similar composite sphere assemblages (CSA, Hashin (1962)) and composite cylinder assemblages (CCA, Hashin and Rosen (1964)), are not covered within the present discussion, either.

For studying materials that are inhomogeneous at a number of (sufficiently widely spread) length scales (e.g., materials in which well defined clusters of inhomogeneities are present), hierarchical procedures that use homogenization at more than one level are a natural extension of the above concepts. Such multi-scale or sequential homogenization models are the subject of a short discussion in chapter 8.

A final group of models used for studying inhomogeneous materials is the direct numerical simulation (DNS) of microstructured structures or samples. Until recently, approaches describing full inhomogeneous configurations were restricted to models the size of which exceeds the microscale by less than, say, one or two orders of magnitude, see, e.g., Papka and Kyriakides (1994), Silberschmidt and Werner (2001), Luxner et al. (2005) or Tekoğlu et al. (2011). Even though such geometries may appear similar to the ones used in windowing approaches, compare chapter 6, such models by design aim at describing the behavior of small structures rather than that of a material. As a consequence, they are typically subjected to boundary conditions and load cases that are more pertinent to structures (among them bending or indentation loads) than to materials. The effects of these boundary conditions as well as the sample’s size in terms of the characteristic length of the inhomogeneities typically play considerable roles in the mechanical behavior of such inhomogeneous bodies, which is often evaluated in terms of force vs. displacement rather than stress vs. strain curves. Recent improvements in computing power have allowed extending DNS models to larger structures, compare, e.g., Bishop et al. (2015). Arguably, such structural models are not part of the “core tool set” of micromechanics, because they do not involve scale transitions.

1.6 Model Verification and Validation

Micromechanical approaches are aimed at generating predictive models for the behavior of inhomogeneous materials. Obviously, model verification, i.e., monitoring a model’s correct setup and implementation, as well as validation, i.e., assessing the accuracy of a model’s representation of the target material behavior, play important roles in continuum micromechanics.

---

13For certain moduli of composites reinforced by particles or by continuous aligned fibers the Halpin–Tsai equations can be obtained from the estimates of Kerner (1956) and Hermans (1967), respectively. Following the procedures outlined in Halpin and Kardos (1976) these models, together with Hill’s connections, eqns. (1.14), and some minor approximations, yield sets of specific “Halpin–Tsai-parameters” which, however, do not appear to have been used widely in practice. In most cases, instead, the Halpin–Tsai equations were applied in a semi-empirical way to other moduli or to other composite geometries, or were used with other sets of parameters of various provenience.
Model verification in continuum micromechanics requires keeping tabs on the consistency and plausibility of modeling assumptions and results throughout a given study. Beyond this, it is often possible to compare predictions with those of other, unrelated micromechanical methods that pertain to analogous geometrical configurations and employ the same constitutive models and material parameters. Bounding methods, see chapter 3, play an especially important role in this respect. Typically, predictions for the linear elastic (or conduction) behavior can be extracted from models, which may then be compared to appropriate bounds (provided, of course, pertinent bounds are available for the configuration under study). Models that fail to fulfill the Hashin–Shtrikman bounds pertaining to the macroscopic symmetry and microscopic geometrical configuration of the material in the linear regime by more than trivial differences (due to, e.g., roundoff errors), must be viewed as flawed and cannot be trusted. In such verification assessments the value of an estimate relative to the pertinent bounds may provide additional information, stiff inhomogeneities in a compliant matrix giving rise to effective moduli closer to the lower and compliant inhomogeneities in a stiff matrix to estimates closer to the upper bound (Torquato, 1991); analogous relationships hold for conduction problems. Convergence studies involving successively finer discretizations when using a numerical engineering method for evaluating the position dependent stress and strain fields in a volume element also fall under the heading of verification. As a rule of thumb, modeling errors present in the linear range tend to be amplified in more complex nonlinear settings.

The validation of micromechanical models against experimental data is obviously highly desirable, but tends to be tricky due to the considerable number of potential sources of discrepancies — among them the representativeness of the volume elements, the comprehensiveness of the constitutive models of the constituents and the accuracy of the material parameters used with these constitutive models, as well as experimental inaccuracies. Obtaining perfect or near-perfect agreement between measurements and predictions without “tuning” input parameters in general is not the most probable of outcomes, especially when damage and failure modeling is involved. Accordingly, even highly successful validation against experimental data cannot replace model verification as defined above. Specifically, a model giving predictions that are close to experimental results but fall outside the pertinent bounds cannot be treated as verified — after all, the agreement with measurements may be due to canceling errors arising from the above issues. In a similar vein, obtaining responses close experimental results from a model that takes major liberties in terms of the microgeometry constitutes neither verification nor validation.

Obtaining reliable values for material parameters pertinent to the microscale tends to be a major challenge, especially when a model involves damage. A common way of dealing with such problems is doing parameter identification via inverse procedures built around the micromechanical model and available experimental results. When following such a strategy, using the same set of experimental data for both parameter identification and model verification must be avoided. In such a setting “overspecialization” of the parameters to that data set may occur, which makes model and parameters unsuitable for generalization to other situations. In a fairly common scenario of this type the nonlinear

\[14\] If data from measurements do not comply with the pertinent Hashin–Shtrikman bounds the most common reason is inaccurate data for the constituent behavior or volume fractions. Measurement errors may, however, also play a role.
Macroscopic behavior of a composite is to be studied and experimental data are limited to results from uniaxial tensile tests. In such a case even a perfect match of predictions with experimental results is far from guaranteeing that load cases involving other macroscopic triaxialities, e.g., shear, or complex stress trajectories will be described reasonably well by the model.
Chapter 2
Mean-Field Methods

In this chapter mean-field relations are discussed mainly for two-phase materials, extensions to multi-phase materials being fairly straightforward in most cases. Special emphasis is put on effective field methods of the Mori–Tanaka type, which may be viewed as the simplest mean-field approaches to modeling inhomogeneous materials that encompass the full physical range of phase volume fractions. Unless specifically stated otherwise, the material behavior of both reinforcements and matrix is taken to be linear (thermo)elastic, both strains and temperature changes being assumed to be small. Perfect bonding between the constituents is assumed in all cases. There is an extensive body of literature covering mean-field approaches, so that the following treatment is far from complete.

2.1 General Relations between Mean Fields in Thermoelastic Two-Phase Materials

Throughout this report additive decomposition of strains is used. For example, for the case of thermoelastoplastic material behavior the total strain tensor can be accordingly be written as

$$\varepsilon = \varepsilon_{\text{el}} + \varepsilon_{\text{pl}} + \varepsilon_{\text{th}}$$

(2.1)

where $\varepsilon_{\text{el}}$, $\varepsilon_{\text{pl}}$ and $\varepsilon_{\text{th}}$ denote the elastic, plastic and thermal strains, respectively. The strain and stress tensors may be split into volumetric/hydrostatic and deviatoric contributions

$$\varepsilon = \varepsilon_{\text{vol}} + \varepsilon_{\text{dev}} = O_{\text{vol}} \varepsilon + O_{\text{dev}} \varepsilon$$

$$\sigma = \sigma_{\text{hyd}} + \sigma_{\text{dev}} = O_{\text{vol}} \sigma + O_{\text{dev}} \sigma$$

(2.2)

$O_{\text{vol}}$ and $O_{\text{dev}}$ being volumetric and deviatoric projection tensors.

For thermoelastic inhomogeneous materials, the macroscopic stress–strain relations can be written in the form

$$\langle \sigma \rangle = E^* \langle \varepsilon \rangle + \lambda^* \Delta T$$

$$\langle \varepsilon \rangle = C^* \langle \sigma \rangle + \alpha^* \Delta T$$

(2.3)

15Because Eshelby and Mori–Tanaka methods are specifically suited for matrix–inclusion-type microtopologies, the expression "composite" is often used in the present chapter instead of the more general designation "inhomogeneous material".
Here the expression $\alpha^* \Delta T$ corresponds to the macroscopic thermal strain tensor, $\lambda^* = -E^* \alpha^*$ is the macroscopic specific thermal stress tensor (i.e., the overall stress response of a fully constrained material to a purely thermal unit load), and $\Delta T$ stands for the (spatially homogeneous) temperature difference with respect to some (stress-free) reference temperature. The constituents, here a matrix $(\text{m})$ and inhomogeneities $(\text{i})$, are also assumed to behave thermoelastically, so that

$$\langle \sigma \rangle^{(\text{m})} = E^{(\text{m})} \langle \varepsilon \rangle^{(\text{m})} + \lambda^{(\text{m})} \Delta T$$

$$\langle \varepsilon \rangle^{(\text{m})} = C^{(\text{m})} \langle \sigma \rangle^{(\text{m})} + \alpha^{(\text{m})} \Delta T$$

$$\langle \sigma \rangle^{(\text{i})} = E^{(\text{i})} \langle \varepsilon \rangle^{(\text{i})} + \lambda^{(\text{i})} \Delta T$$

$$\langle \varepsilon \rangle^{(\text{i})} = C^{(\text{i})} \langle \sigma \rangle^{(\text{i})} + \alpha^{(\text{i})} \Delta T$$

where the relations $\lambda^{(\text{m})} = -E^{(\text{m})} \alpha^{(\text{m})}$ and $\lambda^{(\text{i})} = -E^{(\text{i})} \alpha^{(\text{i})}$ hold.

From the definition of phase averaging, eqn. 1.16, relations between the phase averaged fields in the form

$$\langle \varepsilon \rangle = \xi \langle \varepsilon \rangle^{(\text{i})} + (1 - \xi) \langle \varepsilon \rangle^{(\text{m})} = \varepsilon^a$$

$$\langle \sigma \rangle = \xi \langle \sigma \rangle^{(\text{i})} + (1 - \xi) \langle \sigma \rangle^{(\text{m})} = \sigma^a$$

follow immediately, where $\xi = V^{(\text{i})} = \Omega^{(\text{i})}/\Omega_s$ stands for the volume fraction of the reinforcements and $1 - \xi = V^{(\text{m})} = \Omega^{(\text{m})}/\Omega_s$ for the volume fraction of the matrix. $\varepsilon^a$ and $\sigma^a$ denote the far field (“applied”) homogeneous stress and strain tensors, respectively, with $\varepsilon^a = C^a \sigma^a$. Perfect interfaces between the phases are assumed in expressing the macroscopic strain of the composite as the weighted sum of the phase averaged strains.

The phase averaged strains and stresses can be related to the overall strains and stresses by the phase averaged strain and stress concentration (or localization) tensors $\bar{A}$, $\bar{B}$, $\bar{\kappa}$, and $\bar{\lambda}$ (Hill, 1963), respectively, which are defined for thermoelastic inhomogeneous materials by the expressions

$$\langle \varepsilon \rangle^{(\text{m})} = \bar{A}^{(\text{m})} \langle \varepsilon \rangle + \bar{\beta}^{(\text{m})} \Delta T$$

$$\langle \sigma \rangle^{(\text{m})} = \bar{B}^{(\text{m})} \langle \sigma \rangle + \bar{\kappa}^{(\text{m})} \Delta T$$

$$\langle \varepsilon \rangle^{(\text{i})} = \bar{A}^{(\text{i})} \langle \varepsilon \rangle + \bar{\beta}^{(\text{i})} \Delta T$$

$$\langle \sigma \rangle^{(\text{i})} = \bar{B}^{(\text{i})} \langle \sigma \rangle + \bar{\kappa}^{(\text{i})} \Delta T$$

compare eqn. 1.15 for the purely elastic case. $\bar{A}$ and $\bar{B}$ are referred to as the mechanical or (elastic) phase stress and strain concentration tensors, respectively, and $\bar{\beta}$ as well as $\bar{\kappa}$ are the corresponding thermal concentration tensors.

By using eqns. (2.5) and (2.6), the strain and stress concentration tensors can be shown to fulfill the relations

$$\xi \bar{A}^{(\text{i})} + (1 - \xi) \bar{A}^{(\text{m})} = \mathbf{I}$$

$$\xi \bar{B}^{(\text{i})} + (1 - \xi) \bar{B}^{(\text{m})} = \mathbf{I}$$

$$\xi \bar{\beta}^{(\text{i})} + (1 - \xi) \bar{\beta}^{(\text{m})} = \mathbf{0}$$

$$\xi \bar{\kappa}^{(\text{i})} + (1 - \xi) \bar{\kappa}^{(\text{m})} = \mathbf{0}$$

where $\mathbf{I}$ stands for the symmetric rank 4 identity tensor and $\mathbf{0}$ for the rank 2 null tensor.

The effective elasticity and compliance tensors of a two-phase composite can be obtained from the properties of the constituents and from the mechanical concentration tensors as

$$E^* = \xi E^{(\text{i})} \bar{A}^{(\text{i})} + (1 - \xi) E^{(\text{m})} \bar{A}^{(\text{m})}$$

$$= E^{(\text{m})} + \xi [E^{(\text{i})} - E^{(\text{m})}] \bar{A}^{(\text{i})} = E^{(\text{i})} + (1 - \xi) [E^{(\text{m})} - E^{(\text{i})}] \bar{A}^{(\text{m})}$$

(2.8)
\[ C^* = \xi C^{(i)} \tilde{B}^{(i)} + (1 - \xi) C^{(m)} \tilde{B}^{(m)} = C^{(m)} + \xi [C^{(i)} - C^{(m)}] \tilde{B}^{(i)} = C^{(i)} + (1 - \xi) [C^{(m)} - C^{(i)}] \tilde{B}^{(m)}, \] (2.9)

compare eqn. (1.11). For multi-phase materials with \( N \) phases \((\nu)\) the equivalents of eqn. (2.7) take the form

\[ \sum_{\nu} \xi^{(\nu)} \bar{A}^{(\nu)} = I \quad \sum_{\nu} \xi^{(\nu)} \bar{B}^{(\nu)} = 0, \] (2.10)

and the effective elastic tensors can be evaluated as

\[ E^* = \sum_{\nu} \xi^{(\nu)} E^{(\nu)} \bar{A}^{(\nu)} \quad C^* = \sum_{\nu} \xi^{(\nu)} C^{(\nu)} \bar{B}^{(\nu)} \] (2.11)

by analogy to eqns. (2.8) and (2.9).

The tensor of effective thermal expansion coefficients, \( \alpha^* \), and the specific thermal stress tensor, \( \lambda^* \), can be related to the thermoelastic phase behavior and the thermal concentration tensors as

\[ \alpha^* = \xi [C^{(i)} \bar{\kappa}^{(i)} + \alpha^{(i)}] + (1 - \xi) [C^{(m)} \bar{\kappa}^{(m)} + \alpha^{(m)}] = \xi \alpha^{(i)} + (1 - \xi) \alpha^{(m)} + (1 - \xi) [C^{(m)} - C^{(i)}] \bar{\kappa}^{(i)} \] (2.12)

\[ \lambda^* = \xi [E^{(i)} \bar{\beta}^{(i)} + \lambda^{(i)}] + (1 - \xi) [E^{(m)} \bar{\beta}^{(m)} + \lambda^{(m)}] = \xi \lambda^{(i)} + (1 - \xi) \lambda^{(m)} + (1 - \xi) [E^{(m)} - E^{(i)}] \bar{\beta}^{(i)} \] (2.13)

The above expressions can be derived by inserting eqns. (2.4) and (2.6) into eqns. (2.5) and comparing with eqns. (2.3). Alternatively, the effective thermal expansion coefficient and specific thermal stress coefficient tensors of multi-phase materials can be obtained as

\[ \alpha^* = \sum_{\nu} \xi^{(\nu)} (\bar{B}^{(\nu)})^T \alpha^{(\nu)} \]
\[ \lambda^* = \sum_{\nu} \xi^{(\nu)} (\bar{A}^{(\nu)})^T \lambda^{(\nu)} \] (2.14)

compare (Mandel, 1965; Levin, 1967), the expression for \( \alpha^* \) being known as the Mandel–Levin formula. If the effective compliance tensor of a two-phase material is known eqn. (2.7) can be inserted into eqn. (2.14) to give the overall coefficients of thermal expansion as

\[ \alpha^* = (C^* - C^{(m)}) (C^{(i)} - C^{(m)})^{-1} \alpha^{(i)} - (C^* - C^{(i)}) (C^{(i)} - C^{(m)})^{-1} \alpha^{(m)} \] (2.15)

The mechanical stress and strain concentration tensors for a given phase are linked by expressions of the type

\[ \bar{A}^{(\nu)} = C^{(\nu)} \tilde{B}^{(\nu)} E^* \quad \text{and} \quad \bar{B}^{(\nu)} = E^{(\nu)} \bar{A}^{(\nu)} C^* \] (2.16)
from which $E^*$ and $C^*$ may be eliminated via eqn. (2.11) to give

$$
\bar{A}^{\phi} = C^{\phi} B^{\phi} \left[ \sum_{\phi} \xi^{\phi} C^{\phi} B^{\phi} \right]^{-1}
$$

and

$$
\bar{B}^{\phi} = E^{\phi} A^{\phi} \left[ \sum_{\phi} \xi^{\phi} E^{\phi} A^{\phi} \right]^{-1},
$$

(2.17)

respectively. By invoking the principle of virtual work additional relations were developed (Benveniste and Dvorak, 1990; Benveniste et al., 1991) which link the thermal strain concentration tensors, $\bar{\beta}^{(p)}$, to the mechanical strain concentration tensors, $\bar{A}^{(p)}$, and the thermal stress concentration tensors, $\bar{\kappa}^{(p)}$, to the mechanical stress concentration tensors, $\bar{B}^{(p)}$, respectively, as

$$
\bar{\beta}^{(m)} = [I - \bar{A}^{(m)}][E^{(i)} - E^{(m)}]^{-1}[\lambda^{(m)} - \lambda^{(i)}]
$$

$$
\bar{\beta}^{(i)} = [I - \bar{A}^{(i)}][E^{(m)} - E^{(i)}]^{-1}[\lambda^{(i)} - \lambda^{(m)}]
$$

$$
\bar{\kappa}^{(m)} = [I - \bar{B}^{(m)}][C^{(i)} - C^{(m)}]^{-1}[\alpha^{(m)} - \alpha^{(i)}]
$$

$$
\bar{\kappa}^{(i)} = [I - \bar{B}^{(i)}][C^{(m)} - C^{(i)}]^{-1}[\alpha^{(i)} - \alpha^{(m)}].
$$

(2.18)

From eqns. (2.9) to (2.18) it is evident that the knowledge of one elastic phase concentration tensor is sufficient for describing the full thermoelastic behavior of a two-phase inhomogeneous material within the mean-field framework. A fair number of additional relations between phase averaged tensors have been given in the literature.

An additional set of concentration tensors, useful for describing inhomogeneous materials with matrix–inclusion topology, are the partial strain and stress concentration tensors, $\bar{T}^{(i)}$ and $\bar{W}^{(i)}$, defined by the expressions

$$
\varepsilon^{(i)} = \bar{T}^{(i)} \varepsilon^{(m)}
$$

$$
\sigma^{(i)} = \bar{W}^{(i)} \sigma^{(m)}
$$

(2.19)

compare, e.g., Dvorak and Benveniste (1992). These partial concentration tensors are linked to the phase concentration tensors by the relations

$$
\bar{A}^{(m)} = [\xi \bar{T}^{(i)} + (1 - \xi)I]^{-1}
$$

$$
\bar{B}^{(m)} = [\xi \bar{W}^{(i)} + (1 - \xi)I]^{-1}
$$

$$
\bar{A}^{(i)} = \bar{T}^{(i)} [\xi \bar{T}^{(i)} + (1 - \xi)I]^{-1}
$$

$$
\bar{B}^{(i)} = \bar{W}^{(i)} [\xi \bar{W}^{(i)} + (1 - \xi)I]^{-1}.
$$

(2.20)

The inverse of the partial inhomogeneity stress concentration tensor, $[\bar{W}^{(i)}]^{-1}$ is sometimes referred to as the “bridging tensor”, see Huang (2000).

Equations (2.3) to (2.18) do not account for temperature dependence of the thermoelastic moduli. For a mean-field framework capable of handling temperature dependent moduli for finite temperature excursions and small strains see, e.g., Boussaa (2011).
2.2 Eshelby Tensor and Dilute Matrix–Inclusion Composites

Eshelby’s Eigenstrain Problem

A large percentage of the mean-field descriptions used in continuum micromechanics of materials have been based on the work of Eshelby, who initially studied the stress and strain distributions in homogeneous media that contain a subregion that spontaneously changes its shape and/or size (“undergoes a transformation”) so that it no longer fits into its previous space in the “parent medium”. Eshelby’s results show that if an elastic homogeneous ellipsoidal inclusion (i.e., an inclusion consisting of the same material as the matrix) in an infinite matrix is subjected to a homogeneous strain $\varepsilon_t$ (called the “stress-free strain”, “unconstrained strain”, “eigenstrain”, or “transformation strain”), the stress and strain states in the constrained inclusion are uniform, i.e., $\sigma^{(i)} = \langle \sigma \rangle^{(i)}$ and $\varepsilon^{(i)} = \langle \varepsilon \rangle^{(i)}$. The uniform strain in the constrained inclusion (the “constrained strain”), $\varepsilon_c$, is related to the stress-free strain $\varepsilon_t$ by the expression (Eshelby, 1957)

$$\varepsilon_c = S \varepsilon_t \quad , \quad (2.21)$$

where $S$ is referred to as the (interior point) Eshelby tensor. For eqn. (2.21) to hold, $\varepsilon_t$ may be any kind of eigenstrain that is uniform over the inclusion (e.g., a thermal strain or a strain due to some phase transformation involving no changes in the elastic constants of the inclusion).

The Eshelby tensor $S$ depends only on the material properties of the “parent medium” (the matrix) and on the specific shape of the ellipsoidal inclusions. Eshelby tensors for inclusions of general shape can be obtained as

$$S(x) = \int_{\Omega^{(0)}} \hat{G}^{(0)}(x - x') E^{(0)} d\Omega' = P^{(0)}(x) E^{(0)} \quad , \quad (2.22)$$

compare, e.g., Kachanov and Sevostianov (2018). Here $\hat{G}^{(0)}$ and $E^{(0)}$ stand for the modified Green’s tensor and the elasticity tensor of the parent medium, respectively, and $P^{(0)}$ is known as the Hill tensor, mean polarization factor tensor or shape tensor. The so-called first Eshelby problem consists in finding solutions for the above integral.

Closed form expressions for the Eshelby tensor of spheroidal inclusions in an isotropic matrix are available as functions of the aspect ratio $a$, see, e.g., Pedersen (1983), Tandon and Weng (1984), Mura (1987) or Clyne and Withers (1993), the formulae resulting for continuous fibers of circular cross-section ($a \to \infty$), spherical inclusions ($a = 1$), and thin circular disks or layers ($a \to 0$) being rather simple. Analogous expressions for the Eshelby tensor have also been reported for spheroidal inclusions in a matrix of transversely isotropic material symmetry (Withers, 1989; Parnell and Calvo-Jurado, 2015), provided the material axes of the matrix are aligned with the orientations of non-spherical inclusions. Instead

---

This “Eshelby property” or “Eshelby uniformity” is limited to inclusions and inhomogeneities of ellipsoidal shape (Lubarda and Markenscoff, 1998; Kang and Milton, 2008). For certain non-dilute periodic arrangements of inclusions, however, non-ellipsoidal shapes can also give rise to homogeneous fields (Liu et al., 2007).
of evaluating the Eshelby tensor for a given configuration, the Hill tensor $P^{(0)}$, may be solved for instead, compare, e.g., Ponte Castañeda (1996) or Masson (2008). For detailed discussion of Eshelby and Hill tensors pertaining to ellipsoidal inclusions see Parnell (2016).

For materials of low elastic symmetry the modified Green’s function tensor $\hat{G}^{(0)}$ as required in eqn. (2.22) is not available. However, the problem can be transformed into a surface integral in Fourier space (Mura, 1987), where the modified Green’s tensors are known explicitly for all symmetries. On this basis, approximations to the Eshelby tensors for general elastic symmetries can be obtained by numerical quadrature, compare Gavazzi and Lagoudas (1990).

Eshelby’s Inhomogeneity Problem

For mean-field descriptions of dilute matrix–inclusion composites, the main interest lies on the stress and strain fields in inhomogeneous inclusions (“inhomogeneities”) that are embedded in a matrix. Such cases can be handled on the basis of Eshelby’s theory for homogeneous inclusions, eqn. (2.21), by introducing the concept of an equivalent homogeneous inclusion. This strategy involves replacing an actual perfectly bonded inhomogeneity, which has different material properties than the matrix and which is subjected to a given unconstrained eigenstrain $\epsilon_t$, with a (fictitious) “equivalent” homogeneous inclusion on which a (fictitious) “equivalent” eigenstrain $\epsilon_\tau$ is made to act. This equivalent eigenstrain must be chosen in such a way that the inhomogeneous inclusion and the equivalent homogeneous inclusion attain the same stress state $\sigma^{(i)}$ and the same constrained strain $\epsilon_c$ (Eshelby, 1957; Withers et al., 1989). When $\sigma^{(i)}$ is expressed in terms of the elastic strain in the inhomogeneity or inclusion, this condition translates into the equality

$$\sigma^{(i)} = E^{(i)}[\epsilon_c - \epsilon_t] = E^{(m)}[\epsilon_c - \epsilon_\tau].$$

Here $\epsilon_c - \epsilon_t$ and $\epsilon_c - \epsilon_\tau$ are the elastic strains in the inhomogeneity and the equivalent homogeneous inclusion, respectively. Obviously, in the general case the stress-free strains will be different for the equivalent inclusion and the real inhomogeneity, $\epsilon_t \neq \epsilon_\tau$. Plugging the result of applying eqn. (2.21) to the equivalent eigenstrain, $\epsilon_c = S\epsilon_\tau$, into eqn. (2.23) leads to the relationship

$$\sigma^{(i)} = E^{(i)}[S\epsilon_\tau - \epsilon_t] = E^{(m)}[S - I]\epsilon_\tau,$$

which can be rearranged to obtain the equivalent eigenstrain as a function of the known stress-free eigenstrain $\epsilon_t$ of the real inclusion as

$$\epsilon_\tau = [(E^{(i)} - E^{(m)})S + E^{(m)}]^{-1}E^{(i)}\epsilon_t.$$ 

This, in turn, allows the stress in the inhomogeneity, $\sigma^{(i)}$, to be expressed as

$$\sigma^{(i)} = E^{(m)}(S - I)[(E^{(i)} - E^{(m)})S + E^{(m)}]^{-1}E^{(i)}\epsilon_t.$$

The concept of the equivalent homogeneous inclusion can be extended to cases where a uniform mechanical strain $\epsilon^a$ or external stress $\sigma^a$ is applied to a system consisting of a perfectly bonded inhomogeneous elastic inclusion in an infinite matrix. Here, the strain in
the inclusion, $\epsilon^{(i)}$, is a superposition of the applied strain and of a term $\epsilon_c$ that accounts for
the constraint effects of the surrounding matrix. Well-known expressions for dilute elastic concentration tensors based on the above considerations were proposed by Hill (1965b) and elaborated by Benveniste (1987). For deriving them, the conditions of equal stresses and strains in the actual inhomogeneity (elasticity tensor $E^{(i)}$) and the equivalent inclusion (elasticity tensor $E^{(m)}$) under an applied far field strain $\sigma^a$ take the form

$$\sigma^{(i)} = E^{(i)}[\epsilon^a + \epsilon_c] = E^{(m)}[\epsilon^a + \epsilon_c - \epsilon_\tau]$$

(2.27)

and

$$\epsilon^{(i)} = \epsilon^a + \epsilon_c = \epsilon^a + S\epsilon_\tau$$

(2.28)

respectively, where eqn. (2.21) is used to describe the constrained strain of the equivalent homogeneous inclusion. On the basis of these relationships the strain in the inhomogeneity can be expressed as

$$\epsilon^{(i)} = [I + S C^{(m)}(E^{(i)} - E^{(m)})]^{-1} \epsilon^a$$

(2.29)

Because the strain in the inhomogeneity is homogeneous, $\epsilon^{(i)} = \langle \epsilon \rangle^{(i)}$, the strain concentration tensor for dilute inhomogeneities follows directly from eqn. (2.29) as

$$\bar{A}^{(i)}_{dil} = [I + S C^{(m)}(E^{(i)} - E^{(m)})]^{-1}$$

(2.30)

By using $\langle \epsilon \rangle^{(i)} = C^{(i)}(\sigma)^{(i)}$ as well as $\epsilon^a \cong C^{(m)}\sigma^a$, the dilute stress concentration tensor for the inhomogeneities is found from eqn. (2.29) as

$$\bar{B}^{(i)}_{dil} = E^{(i)}[I + S C^{(m)}(E^{(i)} - E^{(m)})]^{-1} C^{(m)}$$

$$= [I + E^{(m)}(I - S)(C^{(i)} - C^{(m)})]^{-1}$$

(2.31)

In the above approach the Eshelby tensor always refers to the equivalent homogeneity inclusion and, accordingly, it is independent of the material symmetry and elastic properties of the inhomogeneities.

Dilute thermal concentration tensors can be generated from eqns. (2.30) and (2.31) by using eqns. (2.7) and (2.18). Alternative expressions for dilute mechanical and thermal inhomogeneity concentration tensors were given, e.g., by Mura (1987), Wakashima et al. (1988) and Clyne and Withers (1993). All of the above relations were derived under the assumption that the inhomogeneities are dilutely dispersed in the matrix and thus do not “feel” any effects of their neighbors (i.e., they are loaded by the unperturbed applied stress $\sigma^a$ or applied strain $\epsilon^a$, the so-called dilute or non-interacting case). Accordingly, the resulting inhomogeneity concentration tensors are independent of the reinforcement volume fraction $\xi$. The corresponding matrix concentration tensors can be obtained via eqns. (2.7) and, accordingly, depend on $\xi$.

Following, e.g., Dederichs and Zeller (1973) or Kachanov and Sevostianov (2018) the strain field in an inhomogeneity of general shape is given by

$$\epsilon^{(i)}(x) = \langle \epsilon \rangle^{(0)} + (E^{(i)} - E^{(m)}) \int_{\Omega} \tilde{G}^{(0)}(x - x') \epsilon(x') d\Omega'$$

(2.32)

an equation that pertains to expressions such as eqn. (2.29). Solutions of eqn. (2.22), i.e., Hill’s or Eshelby tensors, in general hold for eqn. (2.32) only if $\epsilon$ is constant within
the inhomogeneity, which is the case only for ellipsoids. The integral equation (2.32) is a Lippmann–Schwinger equation and describes the so-called second Eshelby problem\(^\text{18}\).

The stress and strain fields outside an inhomogeneity or a transformed homogeneous inclusion in an infinite matrix are not uniform on the microscale\(^\text{19}\) (Eshelby, 1959). Within the framework of mean-field approaches, which aim to link the average fields in matrix and inhomogeneities with the overall response of inhomogeneous materials, however, it is only the average matrix stresses and strains that are of interest. For dilute composites, such expressions follow directly by combining eqns. (2.30) and (2.31) with eqn. (2.7). Estimates for the overall elastic and thermal expansion tensors can be obtained in a straightforward way from the dilute concentration tensors \(\bar{A}^{(i)}_{\text{dil}}\) and \(\bar{B}^{(i)}_{\text{dil}}\) by using eqns. (2.8) to (2.18); such solutions are often referred to as the Non-Interacting Approximation (NIA). It must be kept in mind, however, that all mean-field expressions of this type are strictly valid only for vanishingly small inhomogeneity volume fractions and give reasonably dependable results only for \(\xi \ll 0.1\). Finally, it is worth noting that exact analytical solutions are available for the phase averaged fields of coated and multi-layered spherical inhomogeneities, see Hervé and Zaoui (1990) and Hervé and Zaoui (1993), respectively.

2.3 Some Mean-Field Methods for Thermoelastic Composites with Aligned Reinforcements

Models of the overall thermoelastic behavior of composites with reinforcement volume fractions of more than a few percent must explicitly account for interactions between inhomogeneities, i.e., for the effects of all surrounding reinforcements on the stress and strain fields experienced by a given fiber or particle. Within the mean-field framework such interaction effects as well as the concomitant perturbations of the stress and strain fields in the matrix are accounted for in a collective way via approximations that are phase-wise constant. Beyond such “background” effects, interactions between individual reinforcements give rise, on the one hand, to inhomogeneous stress and strain fields within each inhomogeneity (intra-particle and intra-fiber fluctuations in the sense of section 1.3) as well as in the matrix. On the other hand, they cause the levels of the average stresses and strains in individual inhomogeneities to differ, i.e., inter-particle and inter-fiber fluctuations are present. Neither of these interactions and fluctuations can be resolved by mean-field methods.

There are two main groups of mean-field approaches to handling non-dilute inhomogeneity volume fractions, viz., effective field and effective medium methods. Figure 2.1 schematically compares the material and loading configurations underlying non-interacting

\(^{18}\)Eshelby tensors in the strict sense are solutions to the integral (2.22). Tensors obtained by solving the integral equation (2.32) and then using eqn. (2.29) to extract \(S\) coincide with the proper Eshelby tensor only if the inhomogeneity is of ellipsoidal shape.

\(^{19}\)The fields outside a single inclusion can be described via the exterior point Eshelby tensor, see, e.g., Ju and Sun (1999) or Meng et al. (2012). From the (constant) interior point fields and the (position dependent) exterior point fields the stress and strain jumps at the interface between inclusion and matrix can be evaluated.
(NIA, “dilute Eshelby”) models, an effective field scheme (MTM) and two effective medium (self-consistent) methods.

\[ \sigma_{\text{a}}^{(m)} = \bar{A}_{\text{dil}}^{(i)} \langle \epsilon \rangle^{(m)} \]

\[ \sigma_{\text{tot}} = \bar{B}_{\text{dil}}^{(i)} \langle \sigma \rangle^{(m)} \]

Figure 2.1: Schematic comparison of the Eshelby method in the non-interacting approximation (NIA), Mori–Tanaka approaches (MTM), differential schemes (DS) and classical (CSCS) as well as generalized self-consistent (GSCS) schemes.

### 2.3.1 Effective Field Approaches

#### Mori–Tanaka-Type Estimates

One way of introducing collective interactions between inhomogeneities consists of approximating the stresses acting on an inhomogeneity, which may be viewed as perturbation stresses caused by the presence of other inhomogeneities (“image stresses”, “background stresses”) superimposed on the applied far field stress, by an appropriate average matrix stress. Alternatively, an average matrix strain may be introduced on the basis of analogous considerations. This concept of effective fields was introduced by Mossotti (1850) and the idea of combining the concept of an average matrix stress with Eshelby-type equivalent inclusion expressions goes back to Brown and Stobbs (1971) as well as Mori and Tanaka (1973). Effective field approaches of this type may be generically referred to as Mori–Tanaka methods.

Benveniste (1987) based his treatment of Mori–Tanaka approaches on the relationships

\[ \langle \epsilon \rangle^{(i)} = \bar{A}_{\text{dil}}^{(i)} \langle \epsilon \rangle^{(m)} \]

\[ \langle \sigma \rangle^{(i)} = \bar{B}_{\text{dil}}^{(i)} \langle \sigma \rangle^{(m)} \]  

(2.33)

which literally transcribe the above assumptions[^20]. Thus, the methodology developed for dilute inhomogeneities is retained and the interactions with the surrounding inhomogeneities are accounted for by suitably modifying the stresses or strains acting on each inhomogeneity. Equations (2.33) may then be viewed as modifications of eqn. (1.15) in which the macroscopic strain or stress, \( \langle \epsilon \rangle \) or \( \langle \sigma \rangle \), is replaced by the phase averaged matrix strain or stress, \( \langle \epsilon \rangle^{(m)} \) and \( \langle \sigma \rangle^{(m)} \), respectively.

[^20]: Equations (2.33) may be viewed as a special case of the more general effective field relations \( \bar{A}^{(i)} = \bar{A}_{\text{dil}}^{(i)} A^{(m)} \) and \( \bar{B}^{(i)} = \bar{B}_{\text{dil}}^{(i)} B^{(m)} \), which may be used for configurations consisting of some “core” embedded in a matrix. Dilute inhomogeneities trivially fulfill these conditions because \( \bar{A}_{\text{dil}}^{(m)} = I \) and \( \bar{B}_{\text{dil}}^{(m)} = I \).
In a further step suitable expressions for $\langle \varepsilon \rangle^{[\text{n}]}$ and/or $\langle \sigma \rangle^{[\text{n}]}$ must be introduced into the scheme. This can be easily done by inserting eqn. (2.33) into eqn. (2.5), leading to the relations

$$
\langle \varepsilon \rangle^{[\text{n}]} = [(1 - \xi)I + \xi \bar{A}_{\text{dil}}^{(i)}]^{-1} \langle \varepsilon \rangle
$$
$$
\langle \sigma \rangle^{[\text{n}]} = [(1 - \xi)I + \xi \bar{B}_{\text{dil}}^{(i)}]^{-1} \langle \sigma \rangle .
$$

(2.34)

These, in turn, allow the Mori–Tanaka strain and stress concentration tensors for matrix and inhomogeneities to be written in terms of the dilute concentration tensors as

$$
\bar{A}_{\text{MT}}^{[\text{n}]} = [(1 - \xi)I + \xi \bar{A}_{\text{dil}}^{(i)}]^{-1} \bar{A}^{(i)}_{\text{MT}} = \bar{A}_{\text{dil}}^{(i)}[(1 - \xi)I + \xi \bar{A}_{\text{dil}}^{(i)}]^{-1}
$$
$$
\bar{B}_{\text{MT}}^{[\text{n}]} = [(1 - \xi)I + \xi \bar{B}_{\text{dil}}^{(i)}]^{-1} \bar{B}^{(i)}_{\text{MT}} = \bar{B}_{\text{dil}}^{(i)}[(1 - \xi)I + \xi \bar{B}_{\text{dil}}^{(i)}]^{-1}
$$

(2.35)

(Benveniste, 1987). Equations (2.35) may be evaluated with any strain and stress concentration tensors $\bar{A}_{\text{dil}}^{(i)}$ and $\bar{B}_{\text{dil}}^{(i)}$ pertaining to dilute inhomogeneities embedded in a matrix. If, for example, the equivalent inclusion expressions, eqns. (2.30) and (2.31), are employed, the Mori–Tanaka matrix strain and stress concentration tensors for the non-dilute composite take the form

$$
\bar{A}_{\text{MT}}^{[\text{n}]} = \{(1 - \xi)I + \xi [I + SC^{[\text{n}]}(E^{(i)} - E^{[\text{n}])}]^{-1}\}^{-1}
$$
$$
\bar{B}_{\text{MT}}^{[\text{n}]} = \{(1 - \xi)I + \xi [I + SC^{[\text{n}]}(E^{(i)} - E^{[\text{n}])}]^{-1}C^{[\text{n}]}\}^{-1} ,
$$

(2.36)


The Mori–Tanaka approximations to the effective elastic tensors are obtained by inserting eqns. (2.35) and/or (2.36) into eqns. (2.8) and (2.9). By inserting eqns. (2.35) into eqns. (2.8) the Mori–Tanaka estimates for the effective elastic tensors are obtained as

$$
E_{\text{MT}}^{(i)} = \left[\xi E^{(i)}\bar{A}_{\text{dil}}^{(i)} + (1 - \xi)E^{[\text{n}]}\right]\left[\xi \bar{A}_{\text{dil}}^{(i)}I\right]^{-1}
$$
$$
C_{\text{MT}}^{(i)} = \left[\xi C^{(i)}\bar{B}_{\text{dil}}^{(i)} + (1 - \xi)C^{[\text{n}]}\right]\left[\xi \bar{B}_{\text{dil}}^{(i)}I\right]^{-1} .
$$

(2.37)

A number of authors gave different but essentially equivalent Mori–Tanaka-type expressions for the phase concentration tensors and effective thermoelastic tensors of inhomogeneous materials, among them Pedersen (1983), Wakashima et al. (1988), Taya et al. (1991), Pedersen and Withers (1992) as well as Clyne and Withers (1993). Alternatively, a Mori–Tanaka method can be formulated to directly give the macroscopic elasticity tensor as

$$
E_{T}^{(i)} = E^{[\text{n}]} \left\{ I - \xi [(E^{(i)} - E^{[\text{n}]}) (S - \xi (S - I)) + E^{[\text{n}]}]^{-1} [E^{(i)} - E^{[\text{n}]})]^{-1}
$$

(2.38)

(Tandon and Weng, 1984). Because eqn. (2.38) does not explicitly use the compliance tensor of the inhomogeneities, $C^{(i)}$, it can be modified in a straightforward way to describe the macroscopic stiffness of porous materials by setting $E^{(i)} \to 0$, producing the relationship

$$
E_{T,\text{por}} = E^{[\text{n}]} \left\{ I + \frac{\xi}{1 - \xi} (I - S)^{-1} \right\}^{-1} .
$$

(2.39)
which, however, should not be used for void volume fractions that are much in excess of, say, $\xi = 0.25^{21}$.

As is evident from their derivation, Mori–Tanaka-type theories at all volume fractions describe composites consisting of aligned ellipsoidal inhomogeneities embedded in a matrix, i.e., inhomogeneous materials with a distinct matrix–inclusion microtopology. More precisely, it was shown by Ponte Castañeda and Willis (1995) that Mori–Tanaka methods are a special case of Hashin–Shtrikman variational estimates (compare section 2.4) in which the spatial arrangement of the inhomogeneities follows an aligned ellipsoidal distribution, which is characterized by the same aspect ratio as the shape of the inhomogeneities themselves (Hu and Weng, 2000), compare fig. 2.2.

Figure 2.2: Sketch of ellipsoidal inhomogeneities in an aligned ellipsoidally distributed spatial arrangement as used implicitly in Mori–Tanaka-type models ($a=2.0$).

For two-phase composites Mori–Tanaka estimates coincide with one of the Hashin–Shtrikman-type bounds (compare section 3.1) and, for composites reinforced by spherical or aligned spheroidal reinforcements that are stiffer than the matrix, their predictions for the overall elastic moduli are, accordingly, always on the low side (see the comparisons in section 3.5 as well as tables 5.1 to 5.3). For materials containing compliant reinforcements in a stiffer matrix, in contrast, they provide upper estimates. In situations of high elastic contrast Mori–Tanaka approaches tend to under- or overestimate the effective elastic properties by a large margin. For discussions of further issues with respect to the range of validity of Mori–Tanaka theories for elastic inhomogeneous two-phase materials see Christensen et al. (1992).

\footnote{Mori–Tanaka theories implicitly assume that the shape of the inhomogeneities can be described by ellipsoids that maintain their aspect ratio throughout the deformation history. In porous or cellular materials with high void volume fractions deformation at the microscale takes place mainly by bending and buckling of cell walls or struts (Gibson and Ashby, 1988), which implies changes to the shapes of the voids. Such effects cannot be captured by Mori–Tanaka models, which, consequently, tend to overestimate the effective stiffness of cellular materials by far. Within the above constraints Mori–Tanaka methods can account for some effects of fluids contained in the pores, compare, e.g., Kitazono et al. (2003).}

26
For multi-phase materials consisting of a matrix (m) into which a number of aligned inhomogeneity phases (i) are embedded, extended the Mori–Tanaka phase concentration tensors can be given as

\[
\bar{A}_{MT}^{(m)} = \left[ \xi^{(m)} I + \sum_{(i) \neq (m)} \bar{A}_{dil}^{(i)} \right]^{-1}
\]
\[
\bar{B}_{MT}^{(m)} = \left[ \xi^{(m)} I + \sum_{(i) \neq (m)} \bar{B}_{dil}^{(i)} \right]^{-1}
\]

\[
\bar{A}_{MT}^{(i)} = \bar{A}_{dil}^{(i)} \left[ \xi^{(m)} I + \sum_{(i) \neq (m)} \bar{A}_{dil}^{(i)} \right]^{-1}
\]
\[
\bar{B}_{MT}^{(i)} = \bar{B}_{dil}^{(i)} \left[ \xi^{(m)} I + \sum_{(i) \neq (m)} \bar{B}_{dil}^{(i)} \right]^{-1}
\]

and the macroscopic elastic tensors are obtained as

\[
E_{MT}^{(m)} = \left[ \xi^{(m)} E^{(m)} + \sum_{(i) \neq (m)} \xi^{(i)} A_{dil}^{(i)} \right] \left[ \xi^{(m)} I + \sum_{(i) \neq (m)} \xi^{(i)} A_{dil}^{(i)} \right]^{-1}
\]
\[
C_{MT}^{(m)} = \left[ \xi^{(m)} C^{(m)} + \sum_{(i) \neq (m)} \xi^{(i)} B_{dil}^{(i)} \right] \left[ \xi^{(m)} I + \sum_{(i) \neq (m)} \xi^{(i)} B_{dil}^{(i)} \right]^{-1}
\]

Multi-phase Mori–Tanaka methods are of an essentially ad-hoc nature and tend to give rise to non-symmetric effective “elastic tensors” in certain situations\(^{22}\) (Benveniste et al., 1991; Ferrari, 1991).

Mori–Tanaka-type theories can be implemented into computer programs in a straightforward way: Because they are explicit algorithms, all that is required are matrix additions, multiplications, and inversions plus expressions for the Eshelby tensor. Despite their limitations, Mori–Tanaka models provide useful accuracy for the elastic contrasts pertaining to most practically relevant composites. This combination of features makes them important tools for evaluating the stiffness and thermal expansion properties of inhomogeneous materials that show a matrix–inclusion topology with aligned inhomogeneities. Mori–Tanaka-type approaches to describing thermoelastoplastic materials are discussed in section 2.4 and “extended” Mori–Tanaka models for nonaligned reinforcements in section 2.6.

Maxwell Schemes

The first “micromechanical” model to appear in the literature, the scheme of Maxwell (1873) for evaluating the effective conductivities of inhomogeneous materials, can be interpreted in terms of a mean-field method in elasticity, compare Sevostianov and Giraud (2013). It is based on studying a single region of inhomogeneous material that contains a sufficient number of inhomogeneities to be treated as an RVE and that is embedded in a much larger matrix region, compare fig. 2.3. This arrangement is described, on the one hand, by a model incorporating the above set of inhomogeneities (left side of fig. 2.3),

\(^{22}\)This problem may crop up when modeling multi-phase composites containing aligned spheroidal inhomogeneities that show both different material behaviors and different aspect ratios; it also tends to occur when studying composites reinforced by nonaligned inhomogeneities, compare section 2.6. Such unphysical behavior of multi-phase Mori–Tanaka methods (which is not an issue for two-phase composites with aligned reinforcements) is related to the fact that they make implicit assumptions on an “ellipsoidal” arrangement of aligned inhomogeneities, compare fig. 2.2 and the section on Ponte–Willis estimates. For recent discussions of such issues in effective field methods see, e.g., Sevostianov and Kachanov (2014) or Rodin and Weng (2014).
which are treated as dilute, and, on the other hand, by a model in which the same region consists of an (unknown) uniform effective material (right side of fig. 2.3), the overall elastic responses of the two configurations being required to be equal. Analogous approaches were used by Giordano (2003) and, independently of Maxwell’s ideas, by Shen and Yi (2001).

Using eqns. (2.8) and (2.9), summing up over the contributions of the individual inhomogeneities \( q \) in the arrangement on the left, and employing dilute Eshelby expressions for both configurations leads to the relations

\[
E^{(m)} + \sum_{\{q\} \neq \{m\}} \xi^{(q)} (E^{(q)} - E^{(m)}) A_{\text{dil}}^{(q)} = E^{(\infty)} + \xi^d (E^* - E^{(m)}) A^{(\text{dil})}_{\text{dil}}
\]

\[
C^{(m)} + \sum_{\{q\} \neq \{m\}} \xi^{(q)} (C^{(q)} - C^{(m)}) B_{\text{dil}}^{(q)} = C^{(\infty)} + \xi^d (C^* - C^{(m)}) B^{(\text{dil})}_{\text{dil}},
\]

where \( \xi^d \) and \( \xi^{(q)} \) are the volume fractions of the inhomogeneous region and of the individual inhomogeneities, respectively. The superscript \( ^{(\infty)} \) indicates that the dilute concentration tensors pertain to a region of effective material embedded in the matrix. Assuming the shapes of the inhomogeneities \( q \) to be identical, so that \( \sum_{\{q\} \neq \{m\}} \xi^{(q)}/\xi^d \) is the inhomogeneity volume fraction \( \xi \), the two-phase Maxwell estimates for the effective elastic tensors, \( E_{\text{MX}}^* \) and \( C_{\text{MX}}^* \), can be extracted from eqn. (2.42) as

\[
E_{\text{MX}}^* = E^{(m)} + \xi (E^{(i)} - E^{(m)}) \left[ I + (S_i - \xi S_d) C^{(m)} (E^{(i)} - E^{(m)}) \right]^{-1}
\]

\[
C_{\text{MX}}^* = C^{(m)} + \xi (C^{(i)} - C^{(m)}) \left[ I + E^{(m)} [(1 - \xi) I - S_i + \xi S_d] (C^{(i)} - C^{(m)}) \right]^{-1}.
\]

Here \( S_i \) and \( S_d \) are Eshelby tensors that describe the shape of the inhomogeneities and of the inhomogeneous region, respectively. For a recent discussion of Maxwell’s method see Sevostianov et al. (2019).

If the inhomogeneous region is chosen to have the same shape as the inhomogeneities, so that \( S_d = S_i \rightarrow S \), the two-phase Maxwell scheme, eqn. (2.43), coincides with the Mori–Tanaka expressions, eqns. (2.35) and (2.36). In the multi-phase and nonaligned cases, however, the Maxwell and Mori–Tanaka estimates differ. For problems involving multiple phases or nonaligned inhomogeneities the proper choice of the shape of the inhomogeneous region described via \( S_d \) is a non-trivial problem, compare the discussions in Sevostianov and Giraud (2013) and Sevostianov and Kachanov (2014).
Ponte–Willis Estimates

Ponte Castañeda and Willis (1995) proposed a refined effective field scheme for inhomogeneous materials that consist of ellipsoidal arrangements of ellipsoidal inhomogeneities embedded in a matrix. In such systems the spatial correlations of the inhomogeneity arrangement can be described by an Eshelby tensor $S_d$ and the shapes of the inhomogeneities via the Eshelby tensor $S_i$, as in the preceding section. The corresponding phase arrangements can be visualized by extending fig. 2.2 to consist of identical, aligned, non-overlapping “safety ellipsoids” (which describe the ellipsoidal arrangement) that contain aligned or nonaligned ellipsoidal inhomogeneities, the aspect ratios of the safety ellipsoids and the inhomogeneities being, in general, different. The inhomogeneity strain concentration tensors of such microgeometries can be expressed as

$$\bar{A}_{PW}^{(i)} = \left[I + (S_i - \xi S_d)C^{(m)}(E^{(i)} - E^{(m)})\right]^{-1}$$

$$= \bar{A}_{dil}^{(i)}\left[I - \xi S_d C^{(m)}(E^{(i)} - E^{(m)})\bar{A}_{dil}^{(i)}\right]^{-1},$$

(2.44)

from which the macroscopic elasticity tensor follows as

$$\bar{E}_{PW} = \bar{E}^{(m)} + \xi(E^{(i)} - E^{(m)})\left[I + (S_i - \xi S_d)C^{(m)}(E^{(i)} - E^{(m)})\right]^{-1}$$

$$= \bar{E}^{(m)} + \left[\xi(E^{(i)} - E^{(m)})\bar{A}_{dil}^{(i)}\right]^{-1},$$

(2.45)

via eqn. (2.8). This model is known as the Hashin–Shtrikman estimates of Ponte Castañeda and Willis (1995). It is strictly valid as long as the inhomogeneities do not penetrate their respective safety ellipsoids. When both the inhomogeneities and the safety ellipsoids are aligned and identical, $S_i = S_d$, eqn. (2.45) becomes equivalent to a Mori–Tanaka expression, showing that Mori–Tanaka methods are a special case of the Ponte–Willis estimates. Analogous expressions to eqn. (2.45) can be obtained with two-phase Maxwell models, compare eqns. (2.43), and the Interaction Direct Derivative (IDD) approach of Du and Zheng (2002). A further related model is the double inclusion model of Hori and Nemat-Nasser (1993), the connections of which to the Ponte–Willis estimates and a number of other micromechanical schemes are discussed in Hu and Weng (2000).

For multi-phase composites the Ponte–Willis estimates for the elasticity tensor take the form

$$\bar{E}_{PW} = \bar{E}^{(m)} + \left\{\sum_{(i) \neq (m)} \xi^{(i)}(E^{(i)} - E^{(m)})\bar{A}_{dil}^{(i)}\right\}^{-1},$$

(2.46)

direct extension of eqn. (2.45). This expression is formally identical to the result obtained with multi-phase versions of the Maxwell scheme, compare Shen and Yi (2001).

Effective Field Method

A further cluster of effective field models, referred to as the Effective Field Method, compare Kanaun and Levin (1994), is based on integral equation approaches and contains Mori–Tanaka models as a special case. Some methods of this group are closely related to the Maxwell and Ponte–Willis estimates. A version of these approaches, the multiparticle effective field method, see, e.g., Buryachenko (2007), can account for interactions between inhomogeneities in composites with random geometry; it was recently formulated in terms
of general integral equations (Buryachenko, 2014). The resulting models are considerably more complex and more powerful than Mori–Tanaka methods. For a further recent mean field model see Saadat et al. (2015).

### 2.3.2 Effective Medium Approaches

The second group of mean-field estimates for the overall thermomechanical moduli of inhomogeneous materials are effective medium approaches, in which an inhomogeneity or some phase arrangement (often referred to as a kernel) is embedded in the effective material, the properties of which are not known a priori. Figure 2.1 sketches two models of this type, the classical self-consistent and generalized self-consistent (compare section 2.4) schemes.

#### Classical Self-Consistent Estimates

If the kernel consists of just an inhomogeneity, classical (or two-phase) self-consistent schemes (CSCS) are obtained, see, e.g., Hill (1965b). They are based on treating all phases as inhomogeneities embedded in an a priori unknown reference medium 0 and requiring the mean-field approximations to the stress and strain perturbations to vanish, which translates into the conditions

\[
\xi (E^{(i)} - E^{0}) \tilde{A}_{dil}^{(i,0)} + (1 - \xi)(E^{(m)} - E^{0}) \tilde{A}_{dil}^{(m,0)} = 0,
\]

\[
\xi (C^{(i)} - C^{0}) \tilde{B}_{dil}^{(i,0)} + (1 - \xi)(C^{(m)} - C^{0}) \tilde{B}_{dil}^{(m,0)} = 0 .
\]  

(2.47)

Accordingly — in contrast to effective field approximations — no distinction is made between contiguous (matrix) and non-contiguous (inhomogeneity) phases. Identifying the reference medium with the effective material and applying eqns. (2.8) the relationships

\[
E_{SC}^{*} = E^{(m)} + \xi [E^{(i)} - E^{(m)}] \tilde{A}_{dil}^{(i*)} = E^{(m)} + \xi [E^{(i)} - E^{(m)}] [I + S^{(i*)} C^{*}(E^{(i)} - E^{SC})]^{-1},
\]

\[
C_{SC}^{*} = C^{(m)} + \xi [C^{(i)} - C^{(m)}] \tilde{B}_{dil}^{(i*)} = C^{(m)} + \xi [C^{(i)} - C^{(m)}] [I + E^{*}(I - S^{(i*)})(C^{(i)} - C^{*})]^{-1}.
\]  

(2.48)

are obtained, which do not obviously show the symmetry in terms of the phases evident in eqn. (2.47). \( \tilde{A}_{dil}^{(i*)} \) and \( \tilde{B}_{dil}^{(i*)} \) may be obtained, e.g., by rewriting eqns. (2.30) and (2.31) for an inhomogeneity that is embedded in an effective medium instead of the matrix, i.e., \( E^{(m)} \rightarrow E^{*} \) and \( C^{(m)} \rightarrow C^{*} \), so that \( \tilde{A}_{dil}^{(i)} \rightarrow \tilde{A}_{dil}^{(i*)}(E^{*}, C^{*}) \) and \( \tilde{B}_{dil}^{(i)} \rightarrow \tilde{B}_{dil}^{(i*)}(E^{*}, C^{*}) \). \( S^{(i*)} \) stands for the Eshelby tensor of the inhomogeneity embedded in the effective material. Equations (2.48) provide systems of implicit nonlinear equations for the unknown elastic tensors \( E_{SC}^{*} \) and \( C_{SC}^{*} \), describing the behavior of the effective medium. These systems can be solved by self-consistent iterative schemes of the type

\[
E_{SC,n+1} = E^{(m)} + \xi [E^{(i)} - E^{(m)}] [I + S_{n}^{(i*)} C_{n}(E^{(i)} - E_{n})]^{-1},
\]

\[
C_{SC,n+1} = [E_{SC,n+1}]^{-1} .
\]  

(2.49)

---

\( ^{23} \)Within the classification of micromechanical methods given in section 1.5, self-consistent mean-field schemes may also be viewed as analytical embedding approaches.
The Eshelby tensor $\mathbf{S}_n^{(i)}$ in eqn. (2.49) pertains to an inhomogeneity embedded in the $n$-th iteration for the effective medium. Accordingly, it must be recomputed for each iteration.\footnote{For aligned spheroidal but non-spherical, isotropic inhomogeneities in an isotropic matrix the effective medium shows transversely isotropic behavior, which must be accounted for in evaluating the Eshelby tensor $\mathbf{S}_n$.}

For multi-phase composites the classical self-consistent estimate for the effective elasticity tensor takes the form

$$
\mathbf{E}^{\ast}_{\text{SC},n+1} = \sum_{\bar{p}} \mathbf{E}^{\bar{p}} \{ \mathbf{I} + \mathbf{S}^{(p^{\ast})}_n \mathbf{C}_{\text{SC},n} \left[ \mathbf{E}^{\bar{p}} - \mathbf{E}_{\text{SC},n} \right] \}^{-1}
$$

$$
\mathbf{C}^{\ast}_{\text{SC},n+1} = \left( \mathbf{E}^{\ast}_{\text{SC},n+1} \right)^{-1},
$$

(2.50)

where the Eshelby tensor $\mathbf{S}^{(p^{\ast})}_n$ pertains to an ellipsoidal inhomogeneity with a shape characterizing phase $\bar{p}$.\footnote{Classical self-consistent schemes have been shown to correspond to perfectly disordered materials (Kröner, 1978) or self-similar hierarchical materials (Torquato, 2002). Because all phases in expressions such as eqn. (2.50) are treated on an equal footing, the CSCS is sometimes referred to as a “symmetric” scheme.}

For two-phase materials the predictions of the CSCS differ noticeably from those of Mori–Tanaka methods in tending to be close to the Hashin–Shtrikman lower bounds (see section 3) if the volume fraction of the more compliant phase is high and close to the upper bounds it is low (compare figs. 3.2 to 3.10). Generally, two-phase self-consistent schemes are well suited to describing the overall properties of two-phase materials that do not show a matrix–inclusion microtopology at some or all of the volume fractions of interest and are, accordingly, not the method of choice for describing “standard” composite materials. The two-phase CSCS may, however, be used for studying Functionally Graded Materials (FGMs), which tend to show an interpenetrating phase topology for volume fractions around $\xi = 0.5$, with one of the constituents acting as the matrix for $\xi \to 0$ and the other for $\xi \to 1$. Multi-phase versions of the CSCS, such as eqn. (2.50), are important methods for modeling polycrystals. When microgeometries described by the CSCS show a geometrical anisotropy, this anisotropy determines the shapes or aspect ratios of the inhomogeneities describing the constituents in the model. For porous materials classical self-consistent schemes predict a breakdown of the stiffness due to percolation of the pores at $\xi = \frac{1}{2}$ for spherical voids and at $\xi = \frac{1}{3}$ for aligned cylindrical voids (Torquato, 2002).

Because self-consistent schemes are by definition implicit methods, their computational requirements are in general higher than those of Mori–Tanaka-type approaches. Like effective field models, they can form the basis for describing the behavior of nonlinear inhomogeneous materials.

**Differential Schemes**

A further important type of effective medium approach are differential schemes (McLaughlin, 1977; Norris, 1985), which may be envisaged as involving repeated cycles of adding small concentrations of inhomogeneities to a material and then homogenizing. Following...
Hashin (1988) the resulting estimates for the overall elastic tensors can be described by the coupled systems of differential equations

\[
\begin{align*}
\frac{dE_D^i}{d\xi} &= \frac{1}{1-\xi} [E^{(i)} - E^*_D] \tilde{A}_{\text{dil}}^{(i*)} \\
\frac{dC_D^i}{d\xi} &= \frac{1}{1-\xi} [C^{(i)} - C^*_D] \tilde{B}_{\text{dil}}^{(i*)}
\end{align*}
\]

with the initial conditions \( E^*_D = E^{(m)} \) and \( C^*_D = C^{(m)} \), respectively, at \( \xi = 0 \). By analogy to eqn. (2.48) \( \tilde{A}_{\text{dil}}^{(i*)} \) and \( \tilde{B}_{\text{dil}}^{(i*)} \) depend on the current approximations to the effective response, \( E^*_D \) and \( C^*_D \). Equations (2.51) can be conveniently integrated with standard numerical algorithms for initial value problems, e.g., Runge–Kutta methods.

Differential schemes can be related to matrix–inclusion microgeometries with polycrystalline distributions of the sizes of the inhomogeneities, corresponding to the repeated homogenization steps\(^26\). Because of their association with specific microgeometries that are not necessarily typical of “classical composites” (in which reinforcement size distributions usually are rather sharp\(^27\)) and due to their higher mathematical complexity (as compared, e.g., to Mori–Tanaka models) Differential Schemes have seen somewhat limited use in studying the mechanical behavior of composite materials.

### 2.4 Other Analytical Estimates for Elastic Composites with Aligned Reinforcements

For convenience, in this section some analytical methods are presented that are of practical importance but are not necessarily based on the mean-field assumptions. The relations between effective elastic tensors and phase averaged fields given in section 2.1 apply to them, nevertheless.

**Hashin–Shtrikman Estimates**

The stress field in an inhomogeneous material can be expressed in terms of a homogeneous comparison (or reference) medium having an elasticity tensor \( E^0 \) as

\[
\sigma(x) = E(x) \varepsilon(x) = E^0 \varepsilon(x) + \tau^0(x)
\]

where \( \tau^0(x) = (E(x) - E^0) \varepsilon(x) \) is referred to as the stress polarization. Phase averaging of \( \tau^0(x) \) over phase \(^9\) leads to the phase averaged stress polarization

\(^26\)The interpretation of differential schemes of involving the repeated addition of infinitesimal volume fractions of inhomogeneities of increasingly larger size, followed each time by homogenization, is due to Roscoe (1973). Avellaneda (1987) showed that differential schemes can be realized by hierarchical laminates. It was pointed out, however, by Hashin (1988) that the requirements of infinitesimal volume fraction and growing size of the inhomogeneities may lead to contradictions.

\(^27\)It is worth noting, though, that, with the exception of the CS CS for two-dimensional conduction (Torquato and Hyun, 2001), none of the mean-field methods discussed in section 2.3 corresponds to a realization in the form of a monodisperse matrix–inclusion microgeometry.
\[ \langle \tau^0 \rangle \hat{\theta} = [\mathbf{E}^0 - \mathbf{E}^0] \langle \varepsilon \rangle \hat{\theta} = \langle \sigma \rangle \hat{\theta} - \mathbf{E}^0 \langle \varepsilon \rangle \hat{\theta} , \] (2.53)

which can be directly applied within the mean-field framework.

Equation (2.53) can be used as the starting point for deriving general mean-field strain concentration tensors in dependence on the elasticity tensor of a reference material. Following Bornert (2001) these concentration tensors take the form

\[
\begin{align*}
\bar{A}^{(\text{m})}_{\text{HS}} &= [\mathbf{L}^0 + \mathbf{E}^{(\text{m})}]^{-1} \left[ \xi (\mathbf{L}^0 + \mathbf{E}^{(i)})^{-1} + (1 - \xi) (\mathbf{L}^0 + \mathbf{E}^{(\text{m})})^{-1} \right]^{-1} \\
\bar{A}^{(i)}_{\text{HS}} &= [\mathbf{L}^0 + \mathbf{E}^{(i)}]^{-1} \left[ \xi (\mathbf{L}^0 + \mathbf{E}^{(i)})^{-1} + (1 - \xi) (\mathbf{L}^0 + \mathbf{E}^{(\text{m})})^{-1} \right]^{-1} .
\end{align*}
\]

(2.54)

The tensor \( \mathbf{L}^0 \) is known as the overall constraint tensor (Hill, 1965b) or Hill’s influence tensor and is defined as

\[ \mathbf{L}^0 = \mathbf{E}^0 [(\mathbf{S}^0)^{-1} - \mathbf{I}] , \]

(2.55)

where the Eshelby tensor \( \mathbf{S}^0 \) must be evaluated with respect to the comparison material. Equations (2.54) lead to estimates for the overall elastic stiffness of two-phase materials of the form

\[
\begin{align*}
\mathbf{E}^{\text{HS}} &= \mathbf{E}^{(\text{m})} + \xi (\mathbf{E}^{(i)} - \mathbf{E}^{(\text{m})}) \left[ \mathbf{I} + (1 - \xi) (\mathbf{L}^0 + \mathbf{E}^{(\text{m})})^{-1} (\mathbf{E}^{(i)} - \mathbf{E}^{(\text{m})}) \right]^{-1} .
\end{align*}
\]

(2.56)

which have been referred to in the literature as Hashin–Shtrikman elasticity tensors (Bornert, 2001).

Standard mean-field models can be obtained as special cases of Hashin–Shtrikman methods by appropriate choices of the comparison material. For example, using the matrix as the comparison material results in Mori–Tanaka methods, whereas the selection of the effective material as comparison material leads to classical self-consistent schemes.

**Generalized Self-Consistent Estimates**

The above Hashin–Shtrikman formalism can be extended in a number of ways, one of which consists in applying it not only to inhomogeneities but also to more complex geometrical entities, called patterns or motifs, embedded in a matrix, see Bornert (1996) and Bornert (2001). Typically the stress and strain fields in such patterns are inhomogeneous even in the dilute case and numerical methods may have to be used for evaluating the corresponding overall constraint tensors. The simplest example of such a pattern consists of a spherical particle or cylindrical fiber surrounded by a constant-thickness layer of matrix at the volume fraction appropriate for the problem. Embedding this motif in the effective material, leads to the three-phase or generalized self-consistent scheme (GSCS), compare fig. 2.1.

Originally, the GCS expressions for the overall elastic moduli were obtained by considering the differential equations describing the elastic response of three-phase regions under appropriate boundary and loading conditions, compare Christensen and Lo (1979) as well as Christensen and Lo (1986). This approach has been applied to isotropic inhomogeneities of spherical or cylindrical shape, where it gives rise to third-order equations for the effective shear modulus \( G^* \) or transverse shear modulus \( G^*_T \), respectively. The results for the bulk
and transverse bulk moduli coincide with those obtained from composite sphere or composite cylinder assemblages, the self-similar microgeometries of which are closely related to, but do not fully correspond to, the GSCS (Christensen, 1998).

Generalized self-consistent schemes may also be obtained via phase-averaged fields evaluated for three-phase systems consisting of coated inhomogeneities embedded in a matrix, see, e.g., Hervé and Zaoui (1990) and Hervé and Zaoui (1995); the latter work provides an extension of the GSCS composites reinforced by transversally isotropic aligned continuous fibers. Extensions of the GSCS to aligned elliptical inhomogeneities in 2D (Huang and Hu, 1995) and to four-phase systems describing materials containing coated inhomogeneities (Dinzart et al., 2016) were also reported.

The three-phase GSCS gives excellent results for inhomogeneous materials with matrix–inclusion topologies and is, accordingly, highly suited to obtaining estimates for the thermoelastic moduli of composite materials reinforced by spherical or equiaxed particles or aligned continuous fibers. Generalized self-consistent schemes are effective medium approaches but are not necessarily mean-field schemes in the sense of section 2.3.

Interpolative Schemes

Interpolative schemes can be constructed that generate estimates between bounds, compare, e.g., Hill (1952), or between mean-field models describing, on the one hand, some composite and, on the other hand, a material in which the roles of matrix and reinforcements are interchanged (referred to, e.g., as Mori–Tanaka and Inverse Mori–Tanaka schemes), see Lielens et al. (1997). Such approaches, while capable of providing useful estimates for some composites, have the drawback of typically not being associated with actual microgeometries.

Three-Point Estimates

A further important group of analytical models for the macroscopic elastic moduli of inhomogeneous materials are estimates based on $n$-point statistics, compare section 3.2. Third-order estimates estimates for the elastic properties of two-phase materials, referred to as “three-point estimates” in what follows, were developed by Torquato (1997, 1998a) on the basis of weak-contrast expansions. They use the same pair of three-point microstructural parameters, $\eta(\xi)$ and $\zeta(\xi)$, as the three-point bounds discussed in section 3.2. These estimates are available for composites reinforced by spherical particles or aligned, cylindrical fibres, lie between the corresponding three-point bounds, and, especially at moderate elastic contrasts and inhomogeneity volume fractions, give excellent analytical predictions for the overall thermoelastic responses of inhomogeneous materials that show the appropriate microgeometries (and are free of flaws).

For further estimates for the overall elastic properties of inhomogeneous materials see, e.g., Torquato (2002).
2.5 Mean-Field Methods for Nonlinear and Inelastic Composites

Since the late 1970s considerable effort has been directed at modeling the mechanical behavior of inhomogeneous materials in which one or more constituents show nonlinear elastic, viscoelastic, elastoplastic or viscoelastoplastic responses. The main motivation of such studies has been the need to describe the time dependent, creep and relaxation behavior of polymer matrix composites and the responses of composites with metallic phases. Mean-field methods have been developed for and successfully adapted to studying many aspects of the above problems.

For nonlinear composites the instantaneous stiffness operators are not phase-wise uniform even if each constituent is homogeneous, so that Eshelby’s results and mean-field models cannot be extended directly from linear elasticity to nonlinear behavior. To deal with this problem, “linear comparison composites” may be defined, which approximate the actual, nonlinear materials’ responses for a given state. Accordingly, the nonlinear problem is reduced to a sequence of linear ones by suitable linearization schemes. In continuum micromechanics, the most important approaches of this type are secant (Tandon and Weng, 1988) and incremental (Hill, 1965a) methods. In addition, tangent concepts (Molinari et al., 1987), affine formulations (Masson et al., 2000; Brenner et al., 2001) and incremental secant algorithms (Wu et al., 2013) have been reported.

2.5.1 Viscoelastic Composites

Viscoelastic materials show hereditary behavior, i.e., their response at a given time depends on their previous load history. Important issues in the mechanical behavior of viscoelastic composites are, on the one hand, quasi-static responses such as relaxation and creep, which can be described via relaxation function and creep compliance tensors. The dynamic behavior under periodic excitations, on the other hand, can be studied via complex modulus tensors.

Correspondence principles are available (Hashin, 1965, 1970) that directly relate the analysis of linear viscoelastic composites to that of linear elastic composites of identical phase geometry for both of the above sets of problems, see also Schapery (1974) and Hashin (1983).

For the quasi-static case the correspondence principle requires formulating the viscoelastic problem in the Laplace–Carson transformed domain, where expressions for the transformed relaxation function and creep compliance tensors are equivalent to estimates for the elasticity and compliance tensors, respectively, in elastic micromechanics. On this basis replacement schemes can be defined (Hashin, 1972) that allow to obtain Laplace–Carson transformed macroscopic moduli, modulus tensors and phase averaged microfields of viscoelastic materials from mean-field results, such as the ones discussed in sections 2.3 to 2.4. The back transformation from the Laplace–Carson to the time domain, however, typically is not straightforward, and closed form solutions are not available in most cases. Accordingly, approximations must be introduced or numerical methods must be used, compare, e.g., Schapery (1962) or Lévesque et al. (2007). The correspondence princi-
ple for periodic excitations uses transforms to Fourier space and the resulting replacement scheme (Hashin, 1972) directly generates effective complex modulus tensors from the effective elastic tensors obtained for a given microgeometry.\[^{28}\]

For in-depth discussions of and alternative concepts for mean-field models of linear and nonlinear viscoelastic composites see, e.g., Paquin et al. (1999), Brenner and Masson (2005), Lévesque et al. (2007), as well as Lahellec and Suquet (2007).

### 2.5.2 (Thermo-)Elastoplastic Composites

The following discussion is restricted to mean-field models based on continuum plasticity.\[^{29}\] Nearly all work reported on such models for elastoplastic or viscoelastoplastic inhomogeneous materials has relied on secant, incremental, tangent, or affine linearization strategies; for an overview see, e.g., Ponte Castañeda and Suquet (1998).

The main difficulties in applying mean-field methods to composites with elastoplastic constituents lie in the path dependence of plastic behavior and in the often strong intraphase fluctuations of the microstress and microstrain fields in elastoplastic inhomogeneous materials. Accordingly, each material point in an elastoplastic phase tends to follow a different trajectory in stress space, so that even a two-phase elastoplastic composite effectively behaves as a multi-phase material and phase averages over reinforcements and matrix are less useful descriptors than in the linear elastic regime. As a consequence, in mean-field models of elastoplastic composites choices have to be made with respect to the linearization procedure, the linear homogenization model, and the phase-wise equivalent stresses and equivalent strains to be used in evaluating the elastoplastic constituent material behavior (Zaoui, 2001).

**Secant Methods for Elastoplastic Composites**

Secant plasticity concepts in continuum micromechanics, see, e.g., Tandon and Weng (1988) or Dunn and Ledbetter (1997), are based on the deformation theory of plasticity, in which the elastoplastic behavior under radial, monotonic loading is approximated by nonlinear elastic models.

In the simplest case of an isotropic elastoplastic phase that is described by $J_2$ plasticity, the secant “elasticity” and “compliance” tensors, $E_{\text{sec}}$ and $C_{\text{sec}}$, take the form

$$E_{\text{sec}} = 3K_{\Theta}^{\Theta} O_{\text{vol}}^E + 2G_{\text{sec}}^{\Theta} O_{\text{dev}}^E \quad \text{and} \quad C_{\text{sec}}^{\Theta} = \frac{1}{3K_{\Theta}^{\Theta}} O_{\text{vol}}^C + \frac{1}{2G_{\text{sec}}^{\Theta}} O_{\text{dev}}^C,$$

\[^{28}\]The correspondence principles can be applied, on the one hand, to analytical expressions, such as the CSA and CCA models (Hashin, 1972) or Mori–Tanaka and self-consistent schemes (Pichler and Lackner, 2009). On the other hand, numerical discrete microfield approaches of the type discussed in chapters 5 to 6 can be adapted to incorporate them, see, e.g., Yi et al. (1998) or Brinson and Lin (1998).

\[^{29}\]In a separate type of model phase averaged stress fields obtained by mean-field methods have been used in dislocation-based descriptions of elastoplastic matrix behavior, see, e.g., Taya and Mori (1987).
respectively. Here \( \mathbf{O}^E \) and \( \mathbf{O}^C \) are the volumetric and deviatoric “partitioning tensors” for elasticities and compliances\(^{30} \), respectively. The secant shear modulus \( G^{\vartheta}_{\text{sec}} \) is equal to the elastic shear modulus of the matrix, \( G^{\vartheta} \), in the elastic range. In the post-yield regime it can be obtained from the current phase averages of the equivalent stress \( \langle \sigma_{\text{equiv}} \rangle^{\vartheta} \) and the equivalent plastic strain \( \langle \varepsilon_{\text{equiv,pl}} \rangle^{\vartheta} \) as

\[
G^{\vartheta}_{\text{sec}} = \frac{G^{\vartheta} \langle \sigma_{\text{equiv}} \rangle^{\vartheta}}{\langle \sigma_{\text{equiv}} \rangle^{\vartheta} + 3G^{\vartheta} \langle \varepsilon_{\text{equiv,pl}} \rangle^{\vartheta}}
\]

on the basis of an additive strain decomposition, compare eqn. (2.1). The equivalent plastic strain \( \langle \varepsilon_{\text{equiv,pl}} \rangle^{\vartheta} \) must be evaluated from \( \langle \sigma_{\text{equiv}} \rangle^{\vartheta} \) via an appropriate hardening law. The bulk moduli \( K^{\vartheta} \) are not affected by yielding due to the \( J_2 \) assumption.

Expressions for the macroscopic secant tensors \( \mathbf{E}^*_{\text{sec}} \) and \( \mathbf{C}^*_{\text{sec}} \) can be obtained from the phase secant tensors, eqn. (2.57), by mean-field relationships equivalent to eqns. (2.8) and (2.9). For the case of elastic inhomogeneities embedded in an elastoplastic matrix these take the form

\[
\begin{align*}
\mathbf{E}^*_{\text{sec}} &= \mathbf{E}^{[m]}_{\text{sec}} + \xi \left( \mathbf{E}^{[i]} - \mathbf{E}^{[m]}_{\text{sec}} \right) \bar{\mathbf{A}}^{[m]}_{\text{sec}} \\
\mathbf{C}^*_{\text{sec}} &= \mathbf{C}^{[m]}_{\text{sec}} + \xi \left( \mathbf{C}^{[i]} - \mathbf{C}^{[m]}_{\text{sec}} \right) \bar{\mathbf{B}}^{[m]}_{\text{sec}}
\end{align*}
\]

The non-dilute secant strain and stress concentration tensors, \( \bar{\mathbf{A}}^{[m]}_{\text{sec}} \) and \( \bar{\mathbf{B}}^{[m]}_{\text{sec}} \), can be generated from dilute secant concentration tensors, \( \bar{\mathbf{A}}^{[m]}_{\text{dil,sec}} \) and \( \bar{\mathbf{B}}^{[m]}_{\text{dil,sec}} \), via a suitable mean-field theory, e.g., a Mori–Tanaka method or a self-consistent scheme, compare section 2.3. The dilute secant concentration tensors and the Eshelby tensors used in them must be evaluated using the current secant tensors of the constituents. Iterative procedures are required for obtaining solutions corresponding to prescribed macroscopic strain or stress states.

In first order (“classical”) methods, the phase average of the equivalent stress required in eqn. (2.58) is approximated from the phase averaged stress tensor,

\[
\langle \sigma_{\text{equiv}} \rangle^{\vartheta} \approx \left[ \frac{3}{2} \langle \sigma \rangle^T_{\text{dev}} \langle \sigma \rangle_{\text{dev}} \right]^{\frac{1}{2}}
\]

This neglects contributions due to the local stress fluctuations and, accordingly, tends to underestimate \( \langle \sigma_{\text{equiv}} \rangle^{\vartheta} \), leading to errors in the estimates for the macroscopic elastoplastic response\(^{31} \). Clear improvements in this respect can be obtained by second order approximations that evaluate the phase averaged equivalent stress in terms of approximations to the second order moments of stress, \( \langle \sigma \otimes \sigma \rangle^{\vartheta} \), (Suquet, 1995; Buryachenko, 1996; Hu, 1997; Pierard et al., 2007), on the basis of energy considerations (Qiu and Weng, 1992), or from a “current stress norm” (Ju and Sun, 2001).

\(^{30}\) In index notation the \( \mathbf{O}^E \) and \( \mathbf{O}^C \) are identical and correspond to the volumetric and deviatoric projection tensors \( \mathbf{O} \) defined in eqn. (2.2). If an engineering notation based on shear angles is used, however, there are differences in the “shear terms” of \( \mathbf{O}^E_{\text{dev}}, \mathbf{O}^C_{\text{dev}}, \) and \( \mathbf{O}_{\text{dev}} \).

\(^{31}\) Because the square of the deviatoric stresses is required for evaluating \( \langle \sigma_{\text{equiv}} \rangle^{\vartheta} \), the fluctuations give rise to non-vanishing contributions. As an extreme case, using eqn. (2.60) for evaluating the equivalent stress leads to predictions that materials with spherical reinforcements will not yield under macroscopically hydrostatic loads or unconstrained thermal expansion. This is in contradiction to other results.
Alternatively, secant theories for composites with nonlinear constituents can be obtained from variational principles (Ponte Castañeda, 1991) or they can be formulated in terms of potentials (Bornert and Suquet, 2001), which allows for a concise mathematical presentation. Because secant models treat elastoplastic composites as nonlinearly elastic materials they are limited to strictly monotonic loading and to radial (or approximately radial) trajectories of the phase averaged stresses of the constituents in stress space, which precludes their use as micromechanically based constitutive models or as lower scale models in multi-scale analysis.

“Modified secant models” (Ponte Castañeda and Suquet, 1998) that use second order approximations for $\langle \sigma_{\text{eqv}} \rangle_{(p)}$ have been found to be highly suitable for materials characterization of elastoplastic composites, where they have shown excellent agreement with predictions from multi-particle unit cell models (Segurado et al., 2002a) and experiments. Modified secant models have also proved quite flexible. For example, a method of this type was adapted to incorporate a nonlocal plasticity model for the matrix (Hu et al., 2005) in order to study particle size effects on the macroscopic yield behavior of particle reinforced MMCs.

### Incremental Methods for Elastoplastic Composites

Incremental mean-field methods can be formulated on the basis of strain and stress rate tensors for elastoplastic phases $\langle \varepsilon \rangle_{(p)}$, $d\langle \varepsilon \rangle_{(p)}$ and $d\langle \sigma \rangle_{(p)}$, which can be expressed as

$$
\begin{align*}
    d\langle \varepsilon \rangle_{(p)} &= \bar{A}_{t}^{(p)} d\langle \varepsilon \rangle + \bar{\beta}_{t}^{(p)} dT \\
    d\langle \sigma \rangle_{(p)} &= \bar{B}_{t}^{(p)} d\langle \sigma \rangle + \bar{\kappa}_{t}^{(p)} dT
\end{align*}
$$

by analogy to eqn. (2.6). Here $d\langle \varepsilon \rangle$ stands for the macroscopic strain rate tensor, $d\langle \sigma \rangle$ for the macroscopic stress rate tensor, and $dT$ for a homogeneous temperature rate. $\bar{A}_{t}^{(p)}$, $\bar{\beta}_{t}^{(p)}$, $\bar{B}_{t}^{(p)}$, and $\bar{\kappa}_{t}^{(p)}$ are instantaneous phase averaged strain and stress concentration tensors, respectively. For elastic inhomogeneities embedded in an elastoplastic matrix, the overall instantaneous (tangent) stiffness tensor of the elastoplastic two-phase composite can be written in terms of the phase properties and the instantaneous concentration tensors as

$$
E_{t}^{*} = E^{(i)} + (1 - \xi)[E_{t}^{(m)} - E^{(i)}]A_{t}^{(m)}
$$

$$
= [C^{(i)} + (1 - \xi)[C_{t}^{(m)} - C^{(i)}]B_{t}^{(m)}]^{-1}
$$

expressions that are closely related to eqns. (2.8) and (2.9).

---

32 The condition of radial loading paths in stress space at the constituent level is generally violated, at least to some extent, in the phases of elastoplastic inhomogeneous materials, even for macroscopic loading paths that are perfectly radial (Pettermann, 1997). This behavior is due to changes in the accommodation of the phase stresses and strains in inhomogeneous materials upon yielding of a constituent and (to a much lesser extent) in the strain hardening regime.

33 Analogous expressions can be derived for elastoplastic inhomogeneities in an elastic matrix or, in general, for composites containing any required number of elastoplastic phases.
Using the Mori–Tanaka formalism of Benveniste (1987), the instantaneous matrix concentration tensors can be written as

\[
\bar{A}^{(m)}_t = \{(1 - \xi)I + \xi[I + S_t C^{(m)}_t(E^{(i)} - E^{(m)}_t)^{-1}]^{-1}\}^{-1}
\]

\[
\bar{B}^{(m)}_t = \{(1 - \xi)I + \xi E^{(i)}[I + S_t C^{(m)}_t(E^{(i)} - E^{(m)}_t)^{-1}]^{-1}\}^{-1}, \tag{2.63}
\]

in direct analogy to eqn. (2.36). Expressions for the instantaneous thermal concentration tensors and instantaneous coefficients of thermal expansion can also be derived by analogy to the corresponding thermoelastic relations, e.g., eqns. (2.13) and (2.18). Equations (2.63) employ the instantaneous Eshelby tensor \(S_t\), which depends on the current state of the (elastoplastic) matrix material, \(E^{(m)}_t\). Because the latter tensor typically shows a low symmetry, \(S_t\) must in general be evaluated numerically.

By replacing rates such as \(d\langle \varepsilon \rangle^{(p)}\) and \(d\langle \sigma \rangle^{(p)}\) with the corresponding finite increments, \(\Delta \langle \varepsilon \rangle^{(p)}\) and \(\Delta \langle \sigma \rangle^{(p)}\), respectively, formulations of eqns. (2.61) to (2.63) can be obtained that are suitable for implementation as micromechanically based constitutive models at the integration point level within Finite Element codes. The resulting incremental Mori–Tanaka (IMT) methods make no assumptions on the overall yield surface and the overall flow potential, the effective material behavior being entirely determined by the incremental mean-field equations and the constitutive behavior of the phases. As a consequence, mapping of the stresses onto the yield surface cannot be handled at the level of the homogenized material and radial return mapping must be applied to the matrix at the microscale instead. This, in turn, implies that the constitutive equations describing the overall behavior cannot be integrated directly (as is the case for homogeneous elastoplastic materials), and iterative algorithms are required. For example, Pettermann (1997) used an implicit Euler scheme in an implementation of an incremental Mori–Tanaka method as a user supplied material routine (UMAT) for the Finite Element code ABAQUS (Simulia, Pawtucket, RI). Extended versions of such algorithms can also handle thermal expansion effects\(^{34}\) and temperature dependent material parameters.

Incremental mean-field models of the type discussed above tend to overestimate the macroscopic strain hardening in the post-yield regime to such an extent that their practical applicability is rather limited, especially for matrix dominated deformation modes, compare, e.g., the discussions by Gilormini (1995) and Suquet (1997). Later developments have involved the use of tangent operators that reflect the symmetry of the elastoplastic phase, e.g., “isotropized” operators\(^{35}\) for statistically isotropic materials such as particle reinforced composites (Bornert, 2001) together with algorithmic modifications (Doghri and Ouaar, 2003; Doghri and Friebel, 2005). These improvements have succeeded in markedly reducing the overprediction of the strain hardening behavior by incremental mean-field models such as incremental Mori–Tanaka methods for particle reinforced composites, making them attractive candidates for use at the lower length scale in hierarchical and multi-

\(^{34}\)Elastoplastic inhomogeneous materials such as metal matrix composites typically show a hysteretic thermal expansion response, i.e., the “coefficients of thermal expansion” are not material properties in the strict sense. This dependence of the thermal expansion behavior on the instantaneous mechanical response requires special treatment within the IMT framework, compare Pettermann (1997).

\(^{35}\)For a discussion of a number of issues pertaining to the use of “isotropic” versus “anisotropic” tangent operators for describing macroscopically isotropic, elastoplastic materials see, e.g., Chaboche and Kanouté (2003).
scale models of ductile matrix composites, compare chapter 8. A pragmatic extension of spectral isotropization schemes to fiber reinforced materials was proposed by Selmi et al. (2011) and recent developments aimed at circumventing isotropization were proposed by Brassart et al. (2012), Lahellec and Suquet (2013) and Wu et al. (2013). Modified IMTs were also successfully extended to the large strain regime (Huber et al., 2007). An alternative approach to handling the above problems, based on transforming the elastic Eshelby tensor, was proposed by Peng et al. (2016).

Incremental mean-field models have typically used eqn. (2.60) for evaluating the phase averaged equivalent stresses, which, accordingly, tend to be underestimated. The improved estimators for \( \langle \sigma_{\text{eqv}} \rangle \) used in modified secant models are not suitable for incremental methods, for which an empirical correction was proposed by Delannay et al. (2007). Furthermore, because they assume elastoplastic phases to yield as a whole once the phase averaged equivalent stress exceeds the yield stress of the elastoplastic constituent, mean-field approaches predict sharp transitions from elastic to plastic states instead of the actual, gradual progress of yielded regions at the microscale.

Other mean-field schemes can also be combined with secant or incremental approaches to obtain descriptions for elastoplastic inhomogeneous materials, the most important application having been the use of classical self-consistent schemes for describing the elastoplastic behavior of polycrystalline materials, see, e.g., Hill (1965a), Hutchinson (1970) and Bergeiller and Zaoui (1981). In the case of incremental methods what has to be done is replacing elastic, Eshelby and concentration tensors with the corresponding instantaneous tensors for each elastoplastic phase in expressions such as eqns. (2.48), (2.50), (2.45) or (2.46). The weaknesses and strengths of such procedures are closely related to those discussed above for Mori–Tanaka-based methods. In addition, mean-field approaches have been employed for obtaining estimates of the nonlinear response of inhomogeneous materials due to microscopic damage or to combinations of damage and plasticity, see, e.g., Tohgo and Chou (1996) and Guo et al. (1997).

As an alternative to directly extending mean-field theories into secant or incremental plasticity, they can also be combined with the Transformation Field Analysis of Dvorak (1992) in order to obtain descriptions of the overall behavior of inhomogeneous materials in the plastic range, see, e.g., Plankensteiner (2000). Such approaches may markedly overestimate the strain hardening of elastoplastic composites because they use elastic accommodation of microstresses and strains throughout the loading history. For modifications aimed at improving this behavior of the Transformation Field Analysis see Chaboche et al. (2001).

For comparisons of predictions of a number of analytical modeling approaches for elastoplastic fiber reinforced composites see Wang and Huang (2018).
2.6 Mean-Field Methods for Composites with Non-aligned Reinforcements

The macroscopic symmetry of composites reinforced by nonaligned short fibers in many cases is isotropic (e.g., for random fiber orientations) or transversely isotropic (for fiber arrangements with axisymmetric orientation distributions, e.g., fibers with planar random orientation). However, processing conditions can give rise to a wide range of fiber orientation distributions and thus to lower overall symmetries, compare, e.g., Allen and Lee (1990).

Descriptions of the microgeometries of nonaligned and hybrid matrix–inclusion composites typically make use of the orientation distribution functions (ODFs) and/or aspect ratio or length distribution functions (LDFs) of the reinforcements, both of which can be determined experimentally. Beyond this, additional, qualitative information may be required on the phase geometry to be modeled, as is the case for materials reinforced by short fibers: At elevated fiber volume fractions local domains of (nearly) aligned fibers are typically observed in short fiber reinforced composites, which give rise to a “grain-type” mesostructure, referred to as “aggregate systems” by Eduljee and McCullough (1993). Composites of this type are appropriately modeled via two-step homogenization schemes that involve meso-level averages at the grain level, see, e.g., Camacho et al. (1990) and Pierard et al. (2004). At low to moderate fiber volume fractions, in contrast, the orientations of neighboring fibers are essentially independent within the geometrical constraints of non-penetration. Such “dispersed systems” (Eduljee and McCullough, 1993) can be modeled via single-step mean-field schemes involving configurational averaging procedures, which may encompass aspect ratio averaging in addition to orientational averaging.

Studying the responses of nonaligned composites with dispersed geometries typically involves orientational averaging of tensor valued variables, which, in general, can be done by direct numerical integration, see, e.g., Pettermann et al. (1997), or on the basis of expansions of the ODF in terms of generalized spherical harmonics (Viglin 1961 expansions), compare, e.g., Advani and Tucker (1987). The latter approach can be formulated in a number of ways, e.g., via texture coefficients or texture matrices, compare Siegmund et al. (2004). For a discussion of a number of issues relevant to configurational averaging see Eduljee and McCullough (1993).

In order to obtain mean-field methods for composites reinforced by nonaligned inhomogeneities one may consider tensors $X^{(i)}$ in suitably oriented local coordinate systems described by the Euler angles $\varphi, \psi, \theta$. By transforming the $X^{(i)}$ into the global coordinate system the tensors $X^{(i)\angle}(\varphi, \psi, \theta)$ are obtained, which form the basis for generating orientational averages

$$\langle \langle X^{(i)} \rangle \rangle = \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi X^{(i)\angle}(\varphi, \psi, \theta) \rho(\varphi, \psi, \theta) \, d\varphi \, d\psi \, d\theta .$$  \hspace{1cm} (2.64)

Here $\rho(\varphi, \psi, \theta)$ denotes the orientation distribution function of the reinforcements, which is assumed to be normalized such that

$$\langle \langle 1 \rangle \rangle = \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi \rho(\varphi, \psi, \theta) \, d\varphi \, d\psi \, d\theta = 1 .$$
 Appropriately transforming the dilute inhomogeneity concentration tensors, \( \tilde{A}^{(i)}_{\text{dil}} \) and \( \tilde{B}^{(i)}_{\text{dil}} \), discussed in section 2.2, provides rotated dilute concentration tensors, \( \tilde{A}^{(i)}_{\text{dil}} \) and \( \tilde{B}^{(i)}_{\text{dil}} \), to which the orientational averaging procedure can be applied in order to arrive at orientation averaged dilute concentration tensors, \( \langle \langle \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle \) and \( \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle \), respectively. The latter concentration tensors can be interpreted as describing an “equivalent phase of nonaligned, non-interacting inhomogeneities”. The core statement of the Mori–Tanaka approach, eqn. (2.33), can then be written in the form

\[
\langle \sigma \rangle^{(i)} = \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle \langle \sigma \rangle^{(m)} = \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle \tilde{B}^{(i)}_{\text{MT}} \langle \sigma \rangle = \tilde{B}^{(i)}_{\text{MT}} \langle \sigma \rangle ,
\]

which allows evaluating the stress concentration tensors for the orientation averaged case, \( \tilde{B}^{(i)}_{\text{MT}} \) and \( \tilde{B}^{(i)}_{\text{MT}} \), as

\[
\tilde{B}^{(i)}_{\text{MT}} = [(1 - \xi)I + \xi \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle ]^{-1} \\
\tilde{B}^{(i)}_{\text{MT}} = \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle [(1 - \xi)I + \xi \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle ]^{-1}
\]

by analogy to eqns. (2.35) and (2.36). Equivalent expressions can be given for the strain concentration tensors. On the basis of eqns. (2.41), (2.8) and (2.9) the effective macroscopic elasticity tensors take the form (Benveniste, 1987)

\[
E_{\text{MT}}^* = E^{(m)} + \xi \langle \langle (E^{(i)} - E^{(m)}) \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle [(1 - \xi)I + \xi \langle \langle \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle ]^{-1} \\
C_{\text{MT}}^* = C^{(m)} + \xi \langle \langle (C^{(i)} - C^{(m)}) \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle [(1 - \xi)I + \xi \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle ]^{-1}
\]

If both phases are isotropic, eqns. (2.67) can be simplified to give

\[
E_{\text{MT}}^* = E^{(m)} + \xi \langle \langle (E^{(i)} - E^{(m)}) \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle [(1 - \xi)I + \xi \langle \langle \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle ]^{-1} \\
C_{\text{MT}}^* = C^{(m)} + \xi \langle \langle (C^{(i)} - C^{(m)}) \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle [(1 - \xi)I + \xi \langle \langle \tilde{B}^{(i)}_{\text{dil}} \rangle \rangle ]^{-1}
\]

For this case analytical expressions for \( E_{\text{MT}}^* \) in terms of two order parameters were given by Giordano (2005).

Orientational averaging can also be used with the effective medium approaches discussed in section 2.3.2. For the classical self-consistent scheme, the equivalent of eqn. (2.49) takes the form

\[
E_{\text{SC},n+1} = E^{(m)} + \xi \langle \langle (E^{(i)} - E^{(m)}) \tilde{A}^{(i)}_{\text{dil},n} \rangle \rangle \\
C_{\text{SC},n+1} = [E_{\text{SC},n+1}]^{-1}
\]

and for the differential scheme the expression

\[
\frac{dE_{\text{D}}^*}{d\xi} = \frac{1}{1 - \xi} \langle \langle (E^{(i)} - E_{\text{D}}^*) \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle
\]

can be obtained from eqn. (2.51). In contrast to the Mori–Tanaka model, which is explicit, multiple evaluations of orientation averages of the type \( \langle \langle (E^{(i)} - E^*) \tilde{A}^{(i)}_{\text{dil}} \rangle \rangle \) are required for both of the above methods. Analytical evaluation of the Eshelby tensors is only supported
if the effective medium is isotropic at all steps, which implies spatially random fiber orientations.

The stress $\langle \sigma \rangle^{(i)}$ evaluated from eqn. (2.65) is an average over all inhomogeneities, irrespective of their orientations, and, accordingly, provides rather limited information. The average stresses in inhomogeneities of a given orientation $(\varphi, \psi, \theta)$, which may be of higher practical interest, can be obtained as

$$\langle \sigma \rangle^{(i)} = \overline{B}_d^{(i)} \overline{B}_M^{(a)} \langle \sigma \rangle,$$

compare (Duschlbauer et al., 2003b). Results obtained with the above relation are in good agreement with numerical predictions for moderate reinforcement volume fractions and elastic contrasts as well as spheroidal fibers of moderate aspect ratios, compare fig. 2.4.

Figure 2.4: Dependence of the averaged stresses in individual fibers on the fiber orientation angle $\Theta$ predicted for a uniaxially loaded SiC/Al MMC reinforced by randomly oriented spheroidal fibers of aspect ratio $a = 5$ (Duschlbauer et al., 2003b). Results for the maximum principal stress $\sigma_1$ and the axial stress $\sigma_{ax}$ obtained with an “extended” Mori–Tanaka scheme are shown as solid and dashed lines, respectively. Unit cell results are presented in terms of the mean values (solid circles) and standard deviations (error bars) of the maximum principal stress within individual fibers. $\Theta$ describes the angle between a given fiber and the loading direction.

“Extended” Mori–Tanaka methods for modeling the elastic behavior of microstructures that contain nonaligned inhomogeneities were developed by a number of authors (Benveniste, 1990; Dunn and Ledbetter, 1997; Pettermann et al., 1997; Mlekusch, 1999). These models differ mainly in the algorithms employed for orientational or configurational

36 The expressions of Clyne and Withers (1993) for Eshelby tensors of spheroids embedded in transversally isotropic matrices hold only for the case that the major axis of the former is aligned with the axis of symmetry of the latter, a condition that is not fulfilled for nontrivial ODFs. The same restriction also holds for Mori–Tanaka-type methods, such as eqns. (2.66) and (2.67). It can, of course, be circumvented by numerical evaluation of the Eshelby or Hill tensors.
averaging. Methods of this type have found many practical applications. They have also been used for studying elastic fabric-reinforced composites, the orientation of the fiber tows being described by appropriate “equivalent orientations” (Gommers et al., 1998) and for approximating the behavior of composites reinforced by helicoidally twisted fibers (Shi et al., 2004). In cases involving finite numbers of fibers of known orientations, the integrals in eqn. (2.64) degenerate into sums and the model becomes a multi-phase Mori–Tanaka method analogous to eqn. (2.41), compare, e.g., Duschlbauer et al. (2011). Extended Mori–Tanaka methods are subject to similar limitations as other multi-phase Mori–Tanaka approaches, and may lead to non-symmetric “effective stiffness tensors” as mentioned in section 2.3.1.

Mean-field approaches based on orientational averaging and Mori–Tanaka methods have also been employed in studies of the nonlinear behavior of nonaligned composites. Secant plasticity schemes of the above type were used for describing ductile matrix materials (Bhattacharyya and Weng, 1994; Dunn and Ledbetter, 1997) and incremental approaches to modeling composites consisting of an elastoplastic matrix reinforced by nonaligned or random short fibers were proposed by Doghri and Tinel (2005) as well as Lee and Simunovic (2000), with the latter work also accounting for debonding between reinforcements and matrix.

An improved approach to studying composites with nonaligned reinforcements is provided by the Hashin–Shtrikman estimates of Ponte Castañeda and Willis (1995). As discussed in section 2.3.1 this method is based on “ellipsoid-in-ellipsoid” phase arrangements, with different Eshelby tensors pertaining to the ellipsoids describing the two-point correlations of the phase arrangement and to those describing the shapes of the inhomogeneities. Equation (2.46) can be directly extended to nonaligned reinforcements by converting the sum into an orientation average to give

$$E^e_{PW} = E^{m} + \zeta^{(i)} \left[ \langle \langle (E^{(i)} - E^{m}) A^{(i)} \rangle \rangle \right] - S_d C^{m}^{-1}$$

(2.72)

As mentioned before, such Hashin–Shtrikman estimates are rigorous (within the effective field setting) provided none of the (aligned) “safety ellipsoids” described by $S_d$ overlap and none of the (nonaligned) inhomogeneities penetrate outside the associated safety ellipsoids. These requirements result in a limited range of strict applicability in terms of inhomogeneity volume fractions, which can be rather small for markedly prolate or oblate reinforcements that show a considerable degree of misalignment.

For the special case of randomly oriented (“uniform random”) fibers or platelets of a given aspect ratio, orientation averaged dilute strain concentration tensors may be constructed, which are known as Wu tensors (Wu, 1966). They can be combined with Mori–Tanaka methods (Benveniste, 1987) or classical self-consistent schemes (Berryman, 1980) to describe composites with randomly oriented phases of matrix–inclusion and certain interpenetrating topologies; in the former case eqn. (2.68) is recovered. Another mean-field method for such materials, the Kuster and Toksöz (1974) model, essentially is a dilute description applicable to matrix–inclusion topologies and tends to give unphysical results at

---

37 Alternative modified Mori–Tanaka models for nonaligned composites that use spatial averaging of the Eshelby tensors, see, e.g., Johannesson and Pedersen (1998), stand on a weaker mechanical basis.
high reinforcement volume fractions. For discussions on the relationships between some of the above approaches see, e.g., Berryman and Berge (1996) or Hu and Weng (2000). Due to the overall isotropic behavior of composites reinforced by randomly oriented fibers or platelets, their elastic response must comply with the Hashin–Shtrikman bounds for macroscopically isotropic materials (Hashin and Shtrikman, 1963), compare Chapter 3 and see also table 5.2 in section 5.5.

In addition to mean-field and unit cell models (see section 5.5) a number of other methods have been proposed for studying composites with nonaligned reinforcements. Most of them are based on the assumption that the contribution of a given fiber to the overall stiffness and strength depends solely on its orientation with respect to the applied load and on its length, interactions between neighboring fibers being neglected. The paper physics approach (Cox, 1952) and the Fukuda–Kawata theory (Fukuda and Kawata, 1974) are based on summing up stiffness contributions of fibers crossing an arbitrary normal section on the basis of fiber orientation and length distribution functions, compare also Jayaraman and Kortschot (1996). Such theories use shear lag models (Cox, 1952; Fukuda and Chou, 1982) or modifications thereof (Fukuda and Kawata, 1974) for describing the behavior of a single fiber embedded in matrix material, the results being typically given as modified rules of mixtures with fiber direction and fiber length corrections. Laminate analogy approaches, see, e.g., Fu and Lauke (1998), approximate nonaligned reinforcement arrangements by a stack of layers each of which pertains to one fiber orientation and, where appropriate, one fiber length. Like related models that are based on orientational averaging of the elastic tensors corresponding to different fiber orientations (Huang, 2001), as well as the so-called aggregate models (Toll, 1992) and pseudo-grain models (Pierard et al., 2004), laminate analogy approaches use Voigt (strain coupling) approximations. Such schemes can be useful especially for describing composites with planar random fiber orientations (Huang, 2001), and they have also formed the basis for describing the nonlinear and damage behavior of composites reinforced by nonaligned short fibers (Doghri and Tinel, 2005; Kammoun et al., 2011).

2.7 Mean-Field Methods for Non-Ellipsoidal Reinforcements

When non-ellipsoidal, inhomogeneous or coated inclusions are subjected to a homogeneous eigenstrain, the resulting stress and strain fields in them are, in general, inhomogeneous. As a consequence, the interior field Eshelby tensors and dilute inhomogeneity concentration tensors depend on the position within such inhomogeneities. Analytical solutions — pertaining to the first Eshelby problem — are available only for certain inhomogeneity shapes, see, e.g., Mura (1987) and Onaka (2001). For some further shapes volume averaged Eshelby tensors have been reported. They have been proposed for use within approximate mean-field models for composites reinforced by non-ellipsoidal inhomogeneities, see, e.g., Zheng et al. (2006) and Hashemi et al. (2009). Depending on how they were evaluated, such tensors pertain to either the first or the second Eshelby problem, compare section 2.2, but not to both of them.
An elegant approach to handling non-ellipsoidal inhomogeneities within a mean-field framework is provided by the contribution tensor formalism, which was introduced by Horii and Nemat-Nasser (1985), elaborated by Kachanov et al. (1994) for porous materials, extended to composite materials by Sevostianov and Kachanov (1999) as well as Eroshkin and Tsukrov (2005) and further modified by Sevostianov and Kachanov (2011). Contribution tensors are formulated in terms of the differences between the effective compliance and elasticity tensors and the corresponding elastic tensors of the matrix,

\[ H = \frac{1}{\xi}(C^* - C^{(m)}) \quad \text{and} \quad N = \frac{1}{\xi}(E^* - E^{(m)}) \],

respectively. \( H \) is referred to as the compliance contribution tensor and \( N \) as the elasticity (or stiffness) contribution tensor. The effective properties of multi-phase composites with matrix–inclusion topology can be obtained by summing up the pertinent contribution tensors,

\[ C^* = C^{(m)} + \sum_{(i) \neq (m)} \xi^{(i)}H^{(i)} \quad \text{and} \quad E^* = E^{(m)} + \sum_{(i) \neq (m)} \xi^{(i)}N^{(i)}, \]

provided direct interactions between inhomogeneities are neglected. General \( H^{(i)} \) and \( N^{(i)} \) are linked to each other by relations of the type

\[ H^{(i)} = -C^{(m)}N^{(i)}(E^{(m)} + \xi N^{(i)})^{-1} \]
\[ N^{(i)} = -E^{(m)}H^{(i)}(C^{(m)} + \xi H^{(i)})^{-1} \]

and are connected to the inhomogeneity stress and strain concentration tensors by the expressions

\[ H^{(i)} = (C^{(i)} - C^{(m)})B^{(i)} \quad \text{and} \quad N^{(i)} = (E^{(i)} - E^{(m)})A^{(i)}. \]

For ellipsoidal inhomogeneities dilute contribution tensors, \( H_{dil}^{(i)} \) and \( N_{dil}^{(i)} \), can be easily obtained, e.g., by combining eqns. (2.76) and (2.30) or (2.31). Using eqns. (2.73) contribution tensors pertaining to dilute inhomogeneities of general shape can be extracted from any solution for the effective compliance or elasticity tensors, \( E_{dil}^* \) and \( C_{dil}^* \), obtained for suitable dilute matrix–inhomogeneity configurations. For non-ellipsoidal reinforcement shapes or highly anisotropic matrix behavior evaluating \( E_{dil}^* \) or \( C_{dil}^* \) typically involves numerical methods. In such models macrohomogeneous or periodicity boundary conditions may be employed, the wide matrix region surrounding the inhomogeneity making the results rather insensitive to the boundary conditions actually used. For the special case of ellipsoidal inhomogeneities, of course, dilute contribution tensors can be generated with much smaller effort from solutions based on Eshelby tensors and equivalents, e.g., via eqn. (2.76). For dilute contribution tensors eqns. (2.75) take the simple forms

\[ H_{dil}^{(i)} = -C^{(m)}N_{dil}^{(i)}E^{(m)} \quad \text{and} \quad N_{dil}^{(i)} = -E^{(m)}H_{dil}^{(i)}C^{(m)}. \]

The dilute contribution tensors directly give rise to non-interacting solutions analogous to the ones discussed at the end of section 2.2. They also form the basis for effective field and effective medium schemes that are suitable for handling non-dilute inhomogeneity volume fractions and are fully equivalent to the models presented in section 2.3, where
“classical” descriptions employing concentration tensors are used. Within a contribution tensor-based two-phase mean-field framework, the Mori–Tanaka method takes the form

\[
H^{(i)}_{\text{MT}} = H^{(i)}_{\text{dil}} [(1 - \xi) I + \xi (C^{(i)} - C^{(m)})^{-1} H^{(i)}_{\text{dil}}]^{-1} \\
N^{(i)}_{\text{MT}} = N^{(i)}_{\text{dil}} [(1 - \xi) I + \xi (E^{(i)} - E^{(m)})^{-1} N^{(i)}_{\text{dil}}]^{-1},
\]

the classical self-consistent scheme becomes

\[
H^{(i)}_{\text{SC}} = \xi (C^{(i)} - C^{(m)}) (C^{(i)} - C^*)^{-1} H^{(i)}_{\text{dil}} \\
N^{(i)}_{\text{SC}} = \xi (E^{(i)} - E^{(m)}) (E^{(i)} - E^*)^{-1} N^{(i)}_{\text{dil}},
\]

the differential scheme is given by

\[
\frac{dH^{(i)}_{\text{Dil}}}{d\xi} = \frac{1}{(1 - \xi)} H^{(i)}_{\text{dil}} \\
\frac{dN^{(i)}_{\text{Dil}}}{d\xi} = \frac{1}{(1 - \xi)} N^{(i)}_{\text{dil}},
\]

and the Maxwell scheme results as

\[
H^{(i)}_{\text{MX}} = \left[ [H^{(i)}_{\text{dil}}]^{-1} - \xi S_d C^{(m)} \right]^{-1} \\
N^{(i)}_{\text{MX}} = \left[ [N^{(i)}_{\text{dil}}]^{-1} - \xi E^{(m)} (I - S_d) \right]^{-1},
\]

compare Eroshkin and Tsukrov (2005). The \(H^{(i)}_{\text{dil}}\) and \(N^{(i)}_{\text{dil}}\) appearing in eqns. (2.79) and (2.80) can be defined as

\[
H^{(i)}_{\text{dil}} = (C^{(i)} - C^*) \bar{B}^{(i)} \quad \text{and} \quad N^{(i)}_{\text{dil}} = (E^{(i)} - E^*) \bar{A}^{(i)}.
\]

For further discussions of contribution-tensor based mean-field models see, e.g., Sevostianov (2014).

When expressions for \(H^{(i)}_{\text{dil}}\) and \(N^{(i)}_{\text{dil}}\) are evaluated from numerical models via eqn. (2.73), both \(\xi\) and terms such as \(E^* - E^{(m)}\) may become very small, so that considerable care is required due to the potential sensitivity of the results to roundoff and discretization errors, especially in the case of non-convex inhomogeneities, compare, e.g., Sevostianov et al. (2008).

Combining numerically evaluated dilute contribution tensors with mean-field methods according to eqns. (2.78) and (2.81) gives useful approximations to the more accurate results obtained by the computationally much more expensive full field models discussed in chapters 4 to 7. Such approaches are, however, less efficient when combined with implicit micromechanical schemes, e.g., eqns. (2.79) and (2.80), which require multiple evaluations of the replacement tensors.
2.8 Mean-Field Methods for Diffusion-Type Problems

The mathematical descriptions of steady-state thermoelasticity and thermal conduction of heterogeneous materials share many common features and can be attacked using similar techniques. Table 2.1 lists the principal variables of these two groups of problems such that the analogies between them are highlighted. A number of other steady-state diffusion phenomena are mathematically analogous to heat conduction (Hashin, 1983), among them electrical conduction and diffusion of moisture. The differential equations describing antiplane shear in elastic solids, Darcy creep flow in porous media, and equilibrium properties such as overall dielectric constants and magnetic permeabilities are also of the Poisson type and thus mathematically equivalent to diffusive transport problems. For further discussions see Torquato (2002).

Table 2.1: Principal variables in steady state elasticity and heat conduction problems.

<table>
<thead>
<tr>
<th>physical problem</th>
<th>elasticity</th>
<th>thermal conduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>field variable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(potential)</td>
<td>displacement field</td>
<td>temperature field</td>
</tr>
<tr>
<td></td>
<td>$u$ [m]</td>
<td>$T$ [K]</td>
</tr>
<tr>
<td>generalized gradient tensor (intensity)</td>
<td>strain field</td>
<td>temperature gradient field</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon$ [ ]</td>
<td>$d$ [Km$^{-1}$]</td>
</tr>
<tr>
<td>generalized flux tensor</td>
<td>stress field</td>
<td>heat flux field</td>
</tr>
<tr>
<td></td>
<td>$\sigma$ [Pa]</td>
<td>$q$ [Wm$^{-2}$]</td>
</tr>
<tr>
<td>generalized property tensor</td>
<td>elasticity</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td></td>
<td>$E$ [Pa]</td>
<td>$K$ [Wm$^{-1}$K$^{-1}$]</td>
</tr>
<tr>
<td></td>
<td>compliance</td>
<td>thermal resistivity</td>
</tr>
<tr>
<td></td>
<td>$C$ [Pa$^{-1}$]</td>
<td>$R$ [W$^{-1}$mK]</td>
</tr>
</tbody>
</table>

An important difference between elasticity and conduction problems concerns the orders of the tensors involved, which is lower in the latter case. The displacements $u$ are vectors whereas the temperatures $T$ are scalars, stresses $\sigma$ and strains $\varepsilon$ are tensors of order 2, whereas the heat fluxes $q$ and thermal gradients $d$ are (physical) vectors, and the elasticity tensor $E$ as well as its inverse, the compliance tensor $C = E^{-1}$, are of order 4, whereas the conductivity tensor $K$ and its inverse, the resistivity tensor $R = K^{-1}$, are of order 2. The differences in the orders of the tensors directly affect the number of effective parameters required for describing the pertinent material property tensors as well as their symmetry properties (Nye, 1957). For example, cubic geometrical symmetry gives rise to macroscopic cubic symmetry in elasticity (with three independent elastic moduli) but isotropic behavior in thermal conduction.

The phase averages of the temperature gradient $d$ and the heat flux $q$ are defined as

$$
\langle d \rangle = \frac{1}{\Omega_\theta} \int_{\Omega_\theta} d(x) \, d\Omega = \frac{1}{\Omega_\theta} \int_{\Gamma_x} T(x) \, n(x) \, d\Gamma
$$

$$
\langle q \rangle = \frac{1}{\Omega_\theta} \int_{\Omega_\theta} q(x) \, d\Omega = \frac{1}{\Omega_\theta} \int_{\Gamma_x} \times q(x) \, d\Gamma
$$

(2.83)
in analogy to eqns. (1.4) and (1.16). The Hill–Mandel condition in thermal conduction becomes
\[ \langle q^T d \rangle = \int_\Omega q^T(x) \, d(x) \, d\Omega = \langle q \rangle^T\langle d \rangle, \]
(2.84)
in direct analogy to eqn. (1.7).

**Dilute Inhomogeneities**

In diffusion-type problems the effects of dilute inhomogeneous inclusions can be described via the rank 2 depolarization tensor (also referred to as “shape tensor” or “diffusion Eshelby tensor”), \( \mathbf{S} \). This tensor, introduced by Fricke (1924), is directly analogous to the “mechanical” Eshelby tensor, \( \mathbf{S} \), discussed in section 2.2. In the case of spheroidal inhomogeneities embedded in an isotropic matrix the depolarization tensor depends only on the formers’ aspect ratio, see, e.g., Hatta and Taya (1986) and Torquato (2002), whereas for transversally isotropic matrix behavior the latter’s material behavior also comes into play, see, e.g., Giraud et al. (2007). For diffusion problems the Eshelby property takes the form of constant fluxes and constant gradients within dilute ellipsoidal inhomogeneities subjected to uniform far-field thermal loads. Dilute gradient and flux concentration tensors can be obtained by analogy to eqns. (2.30) and (2.31) as
\[
\mathbf{A}^{(i)}_{\text{dil}} = [\mathbf{I} + \mathbf{S}^{(\text{in})} (\mathbf{K}^{(i)} - \mathbf{K}^{(\text{in})})]^{-1}, \\
\mathbf{B}^{(i)}_{\text{dil}} = [\mathbf{I} + \mathbf{K}^{(\text{in})} (\mathbf{I} - \mathbf{S}) (\mathbf{R}^{(i)} - \mathbf{R}^{(\text{in})})]^{-1},
\]
(2.85)
providing the basis for non-interacting approximations applicable to dilute inhomogeneity volume fractions. \( \mathbf{I} \) stands for the rank 2 identity tensor.

Equivalents to many of the general relations for mechanical problems discussed in section 2.1 hold for diffusion-type problems, typical examples being the expressions for the effective conductivity and resistivity tensors in terms of the gradient and flux concentration tensors,
\[
\mathbf{K}^* = \xi \mathbf{K}^{(i)} \tilde{\mathbf{A}}^{(i)} + (1 - \xi) \mathbf{K}^{(\text{in})} \tilde{\mathbf{A}}^{(\text{in})}, \\
\mathbf{R}^* = \xi \mathbf{R}^{(i)} \tilde{\mathbf{B}}^{(i)} + (1 - \xi) \mathbf{R}^{(\text{in})} \tilde{\mathbf{B}}^{(\text{in})},
\]
(2.86)
which are directly related to eqns. (2.8) and (2.9), and the linkages between different concentration tensors,
\[
\tilde{\mathbf{A}}^{(g)} = \mathbf{R}^{(g)} \tilde{\mathbf{B}}^{(g)} \mathbf{K}^* \quad \text{and} \quad \tilde{\mathbf{B}}^{(g)} = \mathbf{K}^{(g)} \tilde{\mathbf{A}}^{(g)} \mathbf{R}^*,
\]
(2.87)
which correspond to eqns. (2.16).

**Effective Field Approaches**

For diffusive transport in non-dilute inhomogeneous materials analogous methods to the effective field and effective medium approaches introduced in sections 2.3, 2.4 and 2.6 can be derived, see, e.g., Hatta and Taya (1986), Miloh and Benveniste (1988), Dunn and Taya

49
In the case of Mori–Tanaka methods the phase concentration tensors can be evaluated from their dilute equivalents, eqns. (2.85), as

\[
\begin{align*}
\bar{A}^{(m)}_{MT} & = [(1 - \xi)I + \xi A^{(i)}_{dil}]^{-1} \\
\bar{B}^{(m)}_{MT} & = [(1 - \xi)I + \xi B^{(i)}_{dil}]^{-1}
\end{align*}
\]

in direct analogy to eqns. (2.34) and (2.35). “Extended” Mori–Tanaka models allow to obtain effective conductivity and resistivity tensors as

\[
\begin{align*}
K^{*}_{MT} & = \left[ \xi^{(m)} K^{(m)} + \xi^{(i)} \langle \langle K^{(i)} A^{(i)\dagger} \rangle \rangle \right] \left[ \xi^{(m)} I + \xi^{(i)} \langle \langle A^{(i)\dagger} \rangle \rangle \right]^{-1} \\
R^{*}_{MT} & = \left[ \xi^{(m)} R^{(m)} + \xi^{(i)} \langle \langle R^{(i)} B^{(i)\dagger} \rangle \rangle \right] \left[ \xi^{(m)} I + \xi^{(i)} \langle \langle B^{(i)\dagger} \rangle \rangle \right]^{-1}
\end{align*}
\] (2.89)

Here the non-dilute phase flux concentration tensors are

\[
\langle \langle \bar{B}^{(m)} \rangle \rangle_{MT} = \left[ \xi^{(m)} I + \xi^{(i)} \langle \langle B^{(i)\dagger} \rangle \rangle \right]^{-1}
\]

and \( \bar{B}^{(i)}(\varphi,\psi,\theta) = B^{(i)}_{dil} \langle \langle \bar{B}^{(m)} \rangle \rangle_{MT} \), (2.90)

compare eqns. (2.64) to (2.71), the gradient concentration tensors being obtained by analogy.

The mean-field interpretation of the Maxwell scheme in thermal conduction leads to the relations

\[
\begin{align*}
K^{*}_{MX} & = K^{(m)} + \xi (K^{(i)} - K^{(m)}) \left[ I + (S_{i} - \xi S_{d}) R^{(m)} (K^{(i)} - K^{(m)}) \right]^{-1} \\
R^{*}_{MX} & = \left[ K^{(m)} + \xi (K^{(i)} - K^{(m)}) \right]^{-1} \\
\end{align*}
\] (2.91)

with \( S_{d} \) accounting for the shape of the inhomogeneous region according to fig. 2.3 and \( S_{i} \) for the shape of the inhomogeneities, compare eqn. (2.43). Equation (2.91) also pertains to the conduction equivalent of the Hashin–Shtrikman estimates of Ponte Casta˜neda and Willis (1995), which can alternatively be written as

\[
\begin{align*}
K^{*}_{PW} & = K^{(m)} + [(\xi (K^{(i)} - K^{(m)}) A^{(i)}_{dil})^{-1} - S_{d} R^{(m)}]^{-1} \\
\end{align*}
\] (2.92)

where \( S_{d} \) serves for describing the ellipsoidal arrangement of inhomogeneities by analogy to eqn. (2.45). This scheme was extended to nonaligned inhomogeneities by Duan et al. (2006).

Effective Medium Approaches

The classical self-consistent scheme for thermal conduction, which is often referred to as the symmetrical Bruggemann (1935) method, can be denoted in the form

\[
\begin{align*}
K_{n+1} & = K^{(m)} + \xi [K^{(i)} - K^{(m)}] [I + S_{n}(ps) R_{n}(K^{(i)} - K_{n})]^{-1} \\
R_{n+1} & = [K_{n+1}]^{-1}
\end{align*}
\] (2.93)

which is clearly equivalent to eqn. (2.49). The differential scheme for the effective conductivity tensor can be written as
\[
\frac{dK^*_D}{d\xi} = \frac{1}{1 - \xi} [K^{(i)} - K^{(m)}] A_{\text{dil}}^{(i)} .
\]

For a number of cases analytical solutions are available, see, e.g., Phan-Thien and Pham (2000). Otherwise, the above initial value problem can be solved numerically in analogy to eqn. (2.51).

**Nonaligned Reinforcements**

For conduction problems the equivalents of extended Mori–Tanaka models for nonaligned reinforcements, compare, e.g., eqns. (2.67), take the form

\[
K^*_\text{MT} = K^{(m)} + \xi \langle \langle (K^{(i)} - K^{(m)})A_{\text{dil}}^{(i)} \rangle \rangle [(1 - \xi)I + \xi \langle \langle A_{\text{dil}}^{(i)} \rangle \rangle]^{-1}
\]

\[
R^*_\text{MT} = R^{(m)} + \xi \langle \langle (R^{(i)} - R^{(m)})B_{\text{dil}}^{(i)} \rangle \rangle [(1 - \xi)I + \xi \langle \langle B_{\text{dil}}^{(i)} \rangle \rangle]^{-1} .
\]

Due to the lower ranks of the tensors involved in diffusion-type problems the latter tend to be easier to handle than mechanical ones (for materials that show at least orthotropic symmetry the tensors \(K, R, \tilde{A}, \tilde{B}, S, H\) and \(N\) are diagonal tensors). Extensive discussions of diffusion-type problems in inhomogeneous media can be found, e.g., in Markov (2000), Torquato (2002) and Milton (2002). Analogous mean-field descriptions can also be devised for a number of coupled problems, such as the electromechanical behavior of inhomogeneous materials with at least one piezoelectric constituent, see, e.g., Huang and Yu (1994).

**Non-Ellipsoidal Inhomogeneities and Interfacial Conductances**

Diffusive transport in composites containing non-ellipsoidal inhomogeneities can be described in terms of resistivity and conductivity contribution tensors,

\[
\mathcal{H} = \frac{1}{\xi} (R^* - R^{(m)}) \quad \text{and} \quad \mathcal{N} = \frac{1}{\xi} (K^* - K^{(m)}) ,
\]

by direct analogy to the contribution tensor formalism in elasticity, eqns. (2.73) to (2.81). Dilute contribution tensors can again be obtained from standard mean-field expressions in the case of ellipsoidal inhomogeneities and from numerical models for other reinforcement shapes; in the latter case the linearly independent thermal load cases are required.

The effects of finite interfacial conductances or of interphases (coatings of finite thickness) on the effective conduction behavior of materials reinforced by ellipsoidal fibers or particles can be handled by schemes that employ suitably modified inhomogeneity conductivity tensors within mean-field models of the types discussed in sections 2.85 to 2.95\(^{38}\).
For a discussion of purely analytical equivalent inhomogeneity methods for studying conductivity properties of composites see, e.g., Duan et al. (2007).

If position-dependent interfacial properties or non-ellipsoidal inhomogeneities combined with interfacial effects are involved, dilute inhomogeneity concentration tensors that are averaged over the inhomogeneities may be resorted to (Duschlbauer, 2004). Appropriate averaged “replacement” dilute inhomogeneity concentration tensors, $\bar{A}_{dil}^{(i,r)}$ and $\bar{B}_{dil}^{(i,r)}$, can be evaluated from high-resolution numerical models involving a single inhomogeneity. In conductivity, three linearly independent load cases must be evaluated for a given dilute configuration, e.g., via the Finite Element method. Using eqn. (4.8), the corresponding average gradients and fluxes in the phases can then be extracted, and the replacement concentration tensors evaluated.

In order to achieve a workable formulation, in addition to evaluating $\bar{A}_{dil}^{(i,r)}$ and/or $\bar{B}_{dil}^{(i,r)}$, the “physical” conductivity and resistivity tensors of the inhomogeneities, $K^{(i)}$ and $R^{(i)}$, must be substituted by suitable “replacement” conductivity and resistivity tensors that account for the presence of the interface and are defined as

$$K^{(i,r)} = K^{(m)} + \frac{1}{\xi_{dil}}(K^{*}_{dil} - K^{(m)})[\bar{A}_{dil}^{(i,r)}]^{-1}$$

$$R^{(i,r)} = R^{(m)} + \frac{1}{\xi_{dil}}(R^{*}_{dil} - R^{(m)})[\bar{B}_{dil}^{(i,r)}]^{-1},$$

(2.97)

where $K^{*}_{dil}$ and $R^{*}_{dil}$ are the (numerically evaluated) effective conductivity and resistivity tensors of the dilute single-inhomogeneity configurations and $\xi_{dil}$ is the corresponding reinforcement volume fraction. These replacement conduction tensors ensure consistency within the mean-field framework by enforcing the equivalents of eqns. (2.9) to be fulfilled. The replacement tensors can then be inserted in lieu of the “standard” concentration and conduction tensors into the mean-field expressions given above in order to obtain estimates for composites reinforced by non-ellipsoidal inhomogeneities (Duschlbauer, 2004; Nogales and Böhm, 2008). Analogous replacement tensor schemes can also be set up for the elastic behavior\(^{39}\).

Mean-field models employing replacement tensors following eqn. (2.97), are not suitable for studying non-ellipsoidal inhomogeneities with volume fractions closely approaching unity because in such configurations interfaces between inhomogeneities and matrix cease making physical sense.

In the case of finite interfacial conductances the “equivalent inhomogeneity” or “replacement inhomogeneity” must account for the effects of interfacial temperature jumps, see the discussion by Duschlbauer (2004). Replacement inhomogeneity formalisms can evaluate the effects of such interfacial barrier resistances on the macroscopic conductivity of composites for more complex geometries than “standard methods”, such as the well-known

\(^{39}\)Since concentration tensor based replacement approximations such as eqns. (2.97) do not invoke Eshelby tensors for non-ellipsoidal or coated inhomogeneities, they are not liable to inconsistencies with respect to the first and second Eshelby problems, compare section 2.2.
model of Hasselman and Johnson (1987), an example being polyhedral reinforcements hav- 
ing inhomogeneously distributed interfacial conductances (Nogales and Böhm, 2008).

The above groups of methods — in agreement with numerical models for composites 
with imperfect interfaces (compare section 5.9) and the pertinent bounds (Torquato and 
Rintoul, 1995) — predict that finite interfacial conductances can give rise to marked size ef-
facts in the macroscopic conductivities. Since interfacial resistivities can hardly be avoided, 
below some critical size even highly conductive reinforcements accordingly fail to improve 
the overall conductivity of matrix–inclusion composites of given matrix behavior.

Cross-Property Relationships

Finally, it is worth mentioning that a number of authors have studied cross-property rela-
tions, compare, e.g., Gibiansky and Torquato (1996), and cross-property bounds, see, e.g., 
Sevostianov and Kachanov (2002), that link the elastic and conduction behaviors of given 
microgeometries.
Chapter 3
Bounding Methods

Whereas mean-field methods, unit cell approaches and embedding strategies can typically be used for both homogenization and localization tasks, bounding methods are limited to homogenization. Discussions in this chapter again concentrate on materials consisting of two perfectly bonded constituents.

Rigorous bounds on the overall elastic properties of inhomogeneous materials are typically obtained from appropriate variational (minimum energy) principles. Only outlines of bounding methods are given here; formal treatments were provided, e.g., by Nemat-Nasser and Hori (1993), Bornert (1996), Ponte Castañeda and Suquet (1998), Markov (2000), Torquato (2002) and Milton (2002).

3.1 Classical Bounds

Hill Bounds

Classical expressions for the minimum potential energy and the minimum complementary energy in combination with uniform stress and strain trial functions lead to the simplest variational bounding expressions, the upper bounds of Voigt (1889) and the lower bounds of Reuss (1929). In their tensorial form (Hill, 1952) they are known as the Hill bounds and can be written as

\[
\sum_{(p)} V^{(p)} C^{(p)} - 1 \leq E \leq \sum_{(p)} V^{(p)} E^{(p)}. \quad (3.1)
\]

These bounds, while universal and very simple, do not contain any information on the microgeometry of an inhomogeneous material beyond the phase volume fractions and are too slack for most practical purposes\(^{40}\). However, in contrast to the Hashin–Shtrikman and higher-order bounds, they also hold for volume elements that are too small to be proper RVEs.

\(^{40}\)The bounds on the Young’s moduli obtained from eqn. (3.1) are equivalent to Voigt and Reuss expressions in terms of the corresponding phase moduli only if the constituents have Poisson’s ratios that give rise to equal Poisson contractions. Due to the homogeneous stress and strain assumptions used for obtaining the Hill bounds, the phase strain and stress concentration tensors corresponding to them are \(\bar{A}_V^{(\theta)} = I\) and \(\bar{B}_R^{(\theta)} = I\), respectively.
Hashin–Shtrikman-Type Bounds

Considerably tighter bounds on the macroscopic behavior of inhomogeneous materials can be obtained from a variational formulation due Hashin and Shtrikman (1961). These bounds are formulated in terms of a homogeneous reference material and of stress polarization fields $\tau(x)$ that describe the difference between microscopic stress fields in the inhomogeneous material and in the reference medium, compare eqn. (2.52). In combination with the macrohomogeneity condition, eqn. (1.7), the polarization fields allow the complementary energy of the composite to be expressed in such a way that the “fast” and “slow” contributions are separated, giving rise to the Hashin–Shtrikman variational principle. The Hashin–Shtrikman functional can be written in the form

$$F_{HS}(\tau(x)) = \frac{1}{V} \int_V \{\tau^T(x)[E^0 - E(x)]^{-1} \tau(x) + [\tau(x) - \langle \tau \rangle^*]^T \epsilon'(\tau(x)) + 2\tau^T \epsilon^0\} d\Omega,$$

(3.2)

compare, e.g., Gross and Seelig (2001). The definitions of the variables in eqn. (3.2) follow those in eqn. (2.52). The stationary values of $F_{HS}$ take the form

$$F_{HS} = \epsilon'^0 T (E^* - E^0) \epsilon^0$$

and are maxima when $(E(x) - E^0)$ is positive definite and minima when $(E(x) - E^0)$ is negative definite.

In order to allow an analytical solution for the above variational problem, the highly complex position dependent stress polarizations $\tau(x)$ are approximated by phase-wise constant polarizations $\tau^{q,i}$, compare eqn. (2.53). On this basis the strain fluctuations $\epsilon'(\tau^{q,i})$ can be evaluated by an Eshelby-type eigenstrain procedure. It can be shown that for the case of isotropic constituents and isotropic overall behavior the proper Eshelby tensor to be used is the one for spheres. By optimizing with respect to the $\tau^{q,i}$ the tightest possible bounds within the Hashin–Shtrikman scheme are found.

In the case of two-phase materials with isotropic constituents, in which the matrix is more compliant than the reinforcements, i.e., for $(K^{(i)} - K^{(m)}) > 0$ and $(G^{(i)} - G^{(m)}) > 0$, the above procedure leads to lower and upper bounds for the elasticity tensors of the form

$$E^*_{HS-} = E^{(m)} + \xi \left[(E^{(i)} - E^{(m)})^{-1} + (1 - \xi)S^{(m)}C^{(m)}\right]^{-1} E^{(m)} + \xi(E^{(i)} - E^{(m)}) \bar{A}_{HS-}^{(i)}$$

$$= \left[(1 - \xi)E^{(m)} \bar{A}_{dil}^{(m,m)} + \xi E^{(i)} \bar{A}_{dil}^{(i,m)}\right] \left[(1 - \xi)\bar{A}_{dil}^{(m,m)} + \xi \bar{A}_{dil}^{(i,m)}\right]^{-1},$$

(3.3)

and

$$E^*_{HS+} = E^{(i)} + (1 - \xi) \left[(E^{(m)} - E^{(i)})^{-1} + \xi S^{(i)}C^{(i)}\right]^{-1} E^{(i)} + (1 - \xi)(E^{(m)} - E^{(i)}) \bar{A}_{HS+}^{(i)}$$

$$= \left[(1 - \xi)E^{(m)} \bar{A}_{dil}^{(m,i)} + \xi E^{(i)} \bar{A}_{dil}^{(i,i)}\right] \left[(1 - \xi)\bar{A}_{dil}^{(m,i)} + \xi \bar{A}_{dil}^{(i,i)}\right]^{-1}.\quad(3.4)$$

Here $\bar{A}_{dil}^{(p,q)}$ denotes the dilute strain concentration tensor for phase $(p)$ embedded in phase $(q)$, so that $\bar{A}_{dil}^{(m,m)} = \bar{A}_{dil}^{(i,i)} = I$ and $\bar{A}_{dil}^{(i,m)}$ corresponds to $\bar{A}_{dil}^{(i,i)}$ as defined in eqn. (2.30).
The Hashin–Shtrikman bounds proper, eqns. (3.3) and (3.4) evaluated for spherical inhomogeneities, were stated by Hashin and Shtrikman (1963) as scalar expressions in terms of bounds on the effective bulk modulus $K^*$ and the effective shear modulus $G^*$. They can be realized by the Composite Sphere Assemblage (Hashin, 1962) microstructure and apply to essentially all practically relevant inhomogeneous materials with isotropic phases that show (statistically) isotropic overall symmetry (e.g., composites reinforced by randomly oriented fibers), are statistically homogeneous, and fulfill the condition that the moduli of the constituents are “well ordered”, i.e., $(K^{(i)} - K^{(m)})(G^{(i)} - G^{(m)}) > 0$. The Hashin–Shtrikman formalism recovers the Voigt and Reuss bounds when the reference material is chosen to be much stiffer or much softer, respectively, than either of the constituents. A further set of scalar Hashin–Shtrikman bounds applies to macroscopically transversely isotropic composites that consist of an isotropic matrix reinforced by aligned continuous isotropic fibers (Hashin and Shtrikman, 1962a). An extension of these bounds to composites containing transversely isotropic aligned continuous fibers is given in Hashin (1983).

Walpole (1966) extended the Hashin–Shtrikman bounds by introducing generalized reference materials, thus removing the restriction to well-ordered phase properties. Willis (1977) developed multi-phase Hashin–Shtrikman-type bounds that cover elastically anisotropic macroscopic behavior due to anisotropic constituent behavior and/or aligned ellipsoidal reinforcements. These bounds can be written in a general way as

\[
E_{\text{HSW}}^* = \left[ \sum_{\varrho_l} \xi_{\varrho_l} E^{(\varrho)} \tilde{A}_{\text{dil}}^{(\varrho, l)} \right] - \left[ \sum_{\varrho_l} \xi_{\varrho_l} \tilde{A}_{\text{dil}}^{(\varrho, l)} \right]^{-1}
\]

\[
E_{\text{HSW}}^+ = \left[ \sum_{\varrho_l} \xi_{\varrho_l} E^{(\varrho)} \tilde{A}_{\text{dil}}^{(\varrho, u)} \right] - \left[ \sum_{\varrho_l} \xi_{\varrho_l} \tilde{A}_{\text{dil}}^{(\varrho, u)} \right]^{-1}
\]

(3.5)

Here $\tilde{A}_{\text{dil}}^{(\varrho, l)}$ and $\tilde{A}_{\text{dil}}^{(\varrho, u)}$ are dilute strain concentration tensors that can be evaluated from eqn. (2.30) or an equivalent. They pertain to a phase $\varrho_l$ embedded in soft or stiff reference materials, respectively, described by elasticity tensors $E^l$ and $E^u$. If a “softest” and a “stiffest” constituent cannot be identified unequivocally, suitable fictitious soft and/or stiff reference materials must be constructed for obtaining sharp bounds, see, e.g., the discussions in Parnell and Calvo-Jurado (2015).

41Whereas engineers tend to describe elastic material behavior in terms of Young’s moduli and Poisson’s ratios (which can be measured from uniaxial tensile experiments in a relatively straightforward way), many homogenization expressions are best formulated in terms of bulk and shear moduli, which are directly linked to the hydrostatic and deviatoric responses of materials, compare eqn. (2.57). For obtaining bounds on the effective Young’s modulus $E^*$ from the results on $K^*$ and $G^*$ see Hashin (1983), and for bounding expressions on the Poisson’s ratios $\nu^*$ see Zimmerman (1992). The latter approach can be extended to macroscopically transversely isotropic materials, where, however, it tends to give rise to very slack bounds, especially on the transverse Poisson’s ratios.

42Avellaneda and Milton (1989) constructed hierarchical laminates that are macroscopically isotropic but show overall conductivities falling outside the pertinent Hashin–Shtrikman bounds. Accordingly, macroscopic isotropy by itself is not a sufficient condition for guaranteeing the validity of expressions such as eqns. (3.3) and (3.4). For a discussion of Hashin–Shtrikman-type bounds of macroscopically isotropic composites consisting of anisotropic constituents see, e.g., Nadeau and Ferrari (2001).

43These bounds apply to general (transversely isotropic) two-phase materials, the “in-plane” phase geometries of which are translation invariant in the “out-of-plane” or “axial” direction.
Hashin–Shtrikman-type bounds are not restricted to materials with matrix–inclusion topologies, but hold for any phase arrangement of the appropriate symmetry and phase volume fraction. It is of practical interest that for two-phase materials Mori–Tanaka estimates such as eqn. (2.37) are identical to one of the Hashin–Shtrikman bounds proper given in eqns. (3.3) and (3.4), compare Weng (1990) or Gross and Seelig (2001). The other bound can be obtained by exchanging the roles of inhomogeneities and matrix44 a so-called “color inversion”. Accordingly, for two-phase materials the Hashin–Shtrikman bounds can be evaluated for fairly general aligned phase geometries by simple matrix algebra such as eqn. (2.38). In the case of porous materials, only upper Hashin–Shtrikman-type bounds are available, and only lower bounds can be obtained if one of the constituents is rigid.

Composites reinforced by randomly oriented, isotropic fibers or platelets follow the Hashin–Shtrikman bounds for statistically isotropic materials. A discussion of Hashin–Shtrikman-type bounds on the elastic responses of composites with more general fiber orientation distributions can be found in Eduljee and McCullough (1993). “Extended” Mori–Tanaka models for nonaligned or non-ellipsoidal reinforcements as discussed in sections 2.6 and 2.7 in general are not equivalent to one of the Hashin–Shtrikman bounds.

Hashin–Shtrikman-type bounds are sharp, i.e., they are the tightest bounds that can be given for the kind of geometrical information used (i.e., volume fraction and overall symmetry, corresponding to two-point correlations). Hashin–Shtrikman-type bounds tend to be rather slack, however, i.e., the lower and upper bounds are relatively far apart, especially for elevated phase contrasts.

Hashin–Shtrikman-type variational formulations can also be employed for generating bounds for more general phase arrangements. Evaluating the stress polarizations for “composite regions” consisting of inhomogeneities embedded in matrix gives rise to Hervé–Stolz–Zaoui bounds (Hervé et al., 1991). When complex phase patterns are considered (Bornert, 1996; Bornert et al., 1996) numerical methods must be used for evaluating the polarization fields.

Furthermore, Hashin–Shtrikman-type bounds can be derived for (simple) periodic phase arrangements, see, e.g., Nemat-Nasser and Hori (1993). Among the bounding methods for such phase arrangements are those of Bisegna and Luciano (1996), which uses approximate variational principles evaluated from periodic unit cells via Finite Element models, and of Teply and Dvorak (1988), which evaluates bounds for the elastoplastic behavior of fiber reinforced composites with periodic hexagonal phase arrangements.

44Put more precisely, the lower Hashin–Shtrikman bounds for two-phase materials are obtained from Mori–Tanaka methods when the more compliant material is used as matrix phase, and the upper bounds when it is used as inhomogeneity phase. For matrix–inclusion materials with more than two constituents, Mori–Tanaka expressions are bounds if the matrix is the stiffest or the softest phase, and fall between the appropriate bounds otherwise (Weng, 1990). The Walpole and Willis bounds behave in an analogous way.
3.2 Improved Bounds

A considerable number of statistical descriptors have been used for characterizing the phase arrangements of inhomogeneous materials, see, e.g., Torquato (2002). An important group of them are \( n \)-point probability functions, which are obtained by randomly placing sets on \( n \) points into the microstructure and recording which phases they end up in. For one-point correlations this procedure provides the phase volume fractions. When two-point probabilities are evaluated for two-phase composites, on the one hand, the probabilities of finding both points in the matrix or in the inhomogeneities, \( S_{\text{mm}} \) and \( S_{\text{ii}} \), respectively, can be obtained, compare fig. 3.1. On the other hand, the vector \( \mathbf{r} \) between the two points provides information on the anisotropy of the arrangement and on the dependence of the \( S_{\text{pq}} \) on the distance between the points. The Hill bounds are closely related to one-point and Hashin–Shtrikman bounds to two-point probabilities.

Figure 3.1: Sketch of sampling for one-point, two-point and three-point probability functions in a two-dimensional matrix–inclusion composite; adapted from Gillman et al. (2015).

Three-point probability functions, such as \( S_{\text{iii}}(\mathbf{r}_1, \mathbf{r}_2) \) can provide information on inhomogeneity shapes, sizes and clustering. Most work on three-point correlations has involved isotropic configurations, giving rise to three-point probabilities such as \( S_{\text{iii}}(r_1, r_2, \varphi) \), where \( \varphi \) is the angle between \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \). They form the basis of three-point bounds, which use more complex trial functions than Hashin–Shtrikman bounds and, accordingly, require additional statistical information on the phase arrangement. The resulting “improved bounds” are significantly tighter than the two-point Hashin–Shtrikman-type expressions.

Three-point bounds for statistically isotropic two-phase materials were developed by Beran and Molyneux (1966), Milton (1981) as well as Phan-Thien and Milton (1983) and can be formulated in such a way that the information on the phase arrangement statis-
tics is contained in two three-point microstructural parameters, $\eta(\xi)$ and $\zeta(\xi)$, which take the form of multiple integrals over the three-point correlation functions, see, e.g., Torquato (2002). These correlation functions and the resulting parameters $\zeta(\xi)$ and $\eta(\xi)$ can in principle be obtained for any given two- or three-dimensional microstructure that is statistically homogeneous, but in practice their evaluation may be demanding and require sophisticated numerical algorithms, see Gillman et al. (2015). Analytical expressions or tabulated data for $\zeta(\xi)$ and $\eta(\xi)$ in terms of the reinforcement volume fraction $\xi$ are available for a number of generic microgeometries of practical importance, among them statistically homogeneous isotropic materials containing identical, bidisperse and polydisperse impenetrable (“hard”) spheres (that describe matrix–inclusion composites) as well as monodisperse overlapping spheres (“Boolean models” that can describe many interwoven phase arrangements), and statistically homogeneous transversely isotropic materials reinforced by impenetrable or overlapping aligned cylinders. References to a number of expressions for $\eta$ and $\zeta$ applicable to some two-phase composites can be found in section 3.5, where results from mean-field and bounding approaches are compared. Recently, three-point parameters were published for randomly positioned particles having the shapes of Platonic polyhedra (Gillman et al., 2015). For reviews of higher-order bounds for elastic (as well as other) properties of inhomogeneous materials see, e.g., Quintanilla (1999), Torquato (1991) and Torquato (2002).

Improved bounds can provide highly useful information for low and moderate phase contrasts (as typically found in technical composites), but even they become rather slack for elevated phase contrasts and inhomogeneity volume fractions exceeding, say, $\xi^{(i)} = 0.3$.

### 3.3 Bounds on Nonlinear Mechanical Behavior

Equivalents to the Hill bounds in elasticity, eqn. (3.1), were introduced for nonlinear inhomogeneous materials by Bishop and Hill (1951). For polycrystals the nonlinear equivalents to Voigt and Reuss expressions are usually referred to as Taylor and Sachs bounds, respectively.

In analogy to mean-field estimates for nonlinear material behavior, nonlinear bounds are typically obtained by evaluating sequences of linear bounds. Such bounds typically describe responses to loads that are radial in stress space and usually pertain to uniaxial tensile tests. Talbot and Willis (1985) extended the Hashin–Shtrikman variational principles to obtain one-sided bounds (i.e., upper or lower bounds, depending on the material combination) on the nonlinear mechanical behavior of inhomogeneous materials.

An important development took place with the derivation of a variational principle by Ponte Castañeda (1992), which allows upper bounds on specific stress-strain responses of elastoplastic inhomogeneous materials to be generated on the basis of upper bounds on the elastic tensors.\footnote{The Ponte Castañeda bounds are rigorous for nonlinear elastic inhomogeneous materials and, on the basis of deformation theory, are excellent approximations for materials with at least one elastoplastic constituent. Applying the Ponte Castañeda variational procedure to elastic lower bounds does not necessarily lead to a lower bound for the inelastic behavior.} It uses a sequence of inhomogeneous reference materials, the properties of
which have to be obtained by optimization procedures for each strain level. Essentially, the
variational principle guarantees the best choice for the comparison material at a given load.
The Ponte Castañeda bounds are closely related to mean-field approaches using improved
secant plasticity methods, compare section 2.5. For higher-order bounds on the nonlinear
response of inhomogeneous materials see, e.g., Talbot and Willis (1998).

The study of bounds — like the development of improved estimates — for the overall
nonlinear mechanical behavior of inhomogeneous materials has been an active field of re-
search during the past decades, see the reviews by Suquet (1997), Ponte Castañeda and

3.4 Bounds on Diffusion Properties

Because many bounding approaches are closely related to mean-field models it is not sur-
prising that all of the bounding methods for the macroscopic elastic responses of inho-
mogeneous materials discussed in sections 3.1 and 3.2 have direct equivalents in terms of
diffusion properties by analogy to section 2.8. The equivalents in thermal and electrical
conduction to the elastic Hill bounds, eqn. (3.1), are known as the Wiener (1912) bounds
and take the form

\[ \left( \sum_{(p)} V^{(p)} \mathcal{R}^{(p)} \right)^{-1} \leq \mathcal{K} \leq \sum_{(p)} V^{(p)} \mathcal{K}^{(p)} \]  \tag{3.6}

Hashin–Shtrikman bounds for diffusive transport were developed concurrently with the
bounds for elasticity (Hashin and Shtrikman, 1962b) and can be expressed in analogy to
eqns. (3.3) and (3.4) as

\[ \mathcal{K}_{\text{HS-}}^* = \mathcal{K}^{(m)} + \xi \left( \mathcal{K}^{(i)} - \mathcal{K}^{(m)} \right)^{-1} + (1 - \xi) \mathcal{S}^{(m)} \mathcal{R}^{(m)} \]  \tag{3.7}

Willis bounds can be given in direct equivalence to eqn. (3.5).

Three-point bounds for diffusion properties were proposed, e.g., by Milton (1981). They
require only one of the two statistical parameters used in bounding the elastic behavior,
viz., \( \zeta(\xi) \).

3.5 Comparisons of Mean-Field and Bounding Pre-
dictions for Effective Thermoelastic Moduli

In order to show some of the basic features of the predictions that can be obtained by
different mean-field (and related) approaches and by bounding methods for the thermo-
mechanical responses of inhomogeneous thermoelastic materials, in this section selected
results on the overall elastic moduli and coefficients of thermal expansion are presented as
functions of the reinforcement volume fraction \( \xi \). The comparisons are based on E-glass
particles or fibers embedded in an epoxy matrix, the pertinent material parameters being listed in table 3.1. The elastic contrast of this pair of constituents is $c_{el} \approx 21$ and the thermal expansion contrast takes a value of approximately 0.14.

Table 3.1: Constituent material parameters of the epoxy matrix and the E-glass reinforcements used in generating figs. 3.2 to 3.10.

<table>
<thead>
<tr>
<th></th>
<th>$E$ [GPa]</th>
<th>$\nu$</th>
<th>$\alpha$ [1/K]</th>
</tr>
</thead>
<tbody>
<tr>
<td>matrix</td>
<td>3.5</td>
<td>0.35</td>
<td>$3.6 \times 10^{-5}$</td>
</tr>
<tr>
<td>reinforcements</td>
<td>74.0</td>
<td>0.2</td>
<td>$4.9 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Figures 3.2 and 3.3 show predictions for the overall Young’s and shear moduli of a particle reinforced two-phase composite using the above constituent parameters. The Hill bounds can be seen to be very slack. A macroscopically isotropic two-phase composite being studied, the Mori–Tanaka estimates (MTM) coincide with the lower Hashin–Shtrikman bounds (H/S LB), compare section 3.1. The classical self-consistent scheme (CSCS) shows a typical behavior in that it is close to one Hashin–Shtrikman bound at low volume fractions, approaches the other at high volume fractions, and displays a transitional behavior in the form of a sigmoid curve in-between.

The three-point bounds (3PLB and 3PUB) shown in figs. 3.2 pertain to impenetrable spherical particles of equal size and use expressions for the statistical parameters $\eta$ and $\zeta$ listed by Torquato (2002), which are available for reinforcement volume fractions up to $\xi = 0.64$. As expected, these improved bounds are significantly tighter than the Hashin–

Figure 3.2: Bounds and estimates for the effective Young’s moduli of glass/epoxy particle reinforced composites as functions of the particle volume fraction.

\[^46\]This value approaches the maximum volume fraction achievable in random packings of identical spheres, which is $\xi \approx 0.64$ (Weisstein, 2000).
Figure 3.3: Bounds and estimates for the effective shear moduli of glass/epoxy particle reinforced composites as functions of the particle volume fraction.

Shtrikman bounds. The three-point estimates (3PE) of Torquato (1998a), which were evaluated with the same choice of $\eta$ and $\zeta$, fall between the three-point bounds. For the composite considered here the results from the generalized self-consistent scheme (GSCS), which predicts a slightly stiffer behavior than the Mori–Tanaka method, are very close to the lower three-point bounds even though the GSCS is not associated with monodisperse particle microgeometries. The predictions of the differential scheme (DS) pertain to composites with polydisperse reinforcements and can be seen to be stiffer than either the three-point estimates for identical spheres or the GSCS results.

Alternatively, the elastic moduli of macroscopically isotropic composites can be visualized by plotting the shear modulus over the bulk modulus following Berryman and Milton (1988). In fig. 3.4 this format is used to compare the Hashin–Shtrikman (outermost, solid box) and three-point bounds for impenetrable, identical spheres (inner, dashed box) with a number of estimates for a glass–epoxy composite having a particle volume fraction of $\xi = 0.4$. The typical behavior of composites consisting of a compliant matrix reinforced by stiff particles is shown: the three-point bounds and all estimates pertinent to matrix–inclusion materials cluster in the corner corresponding to low bulk and shear moduli, whereas the classical self-consistent scheme predicts clearly different responses appropriate for materials having other phase-level topologies.

Predictions for the macroscopic coefficients of thermal expansion of this macroscopically isotropic inhomogeneous material are presented in fig. 3.5. Levin’s formula, eqn.(2.14), was combined with the Hashin–Shtrikman bounds, the three-point bounds, the generalized self-consistent estimates and the three-point estimates for the effective bulk modulus to obtain the corresponding bounds and estimates for the CTE. In the case of the classical self-consistent and differential schemes eqns. (2.12) and (2.18) were used. Because they are based on the same estimates for the effective bulk modulus, the results presented for
Figure 3.4: Bounds and estimates for the effective bulk and shear moduli of a glass/epoxy particle reinforced composite of particle volume fraction $\xi^{(p)} = 0.4$.

the GSCS and the Mori–Tanaka-scheme coincide with the upper Levin/Hashin–Shtrikman bounds.

Figure 3.5: Bounds and estimates for the effective CTEs of glass/epoxy particle reinforced composites as functions of the particle volume fraction.

Applying the constituent data given in table 3.1 to an epoxy matrix reinforced by continuous aligned glass fibers gives rise to transversally isotropic macroscopic behavior. Pertinent results are presented in figs. 3.6 to 3.10 for the overall transverse Young’s
moduli\textsuperscript{47}, the overall axial and transverse shear moduli, as well as the overall axial and transverse coefficients of thermal expansion. The results for the three-point bounds shown are based on the formalisms of Silnutzer (1972) and Milton (1981), correspond to a microgeometry of aligned impenetrable circular cylindrical fibers of equal diameter, and use

\textsuperscript{47}The estimates and bounds for the axial Young’s moduli are indistinguishable from each other (and from the rule-of-mixture result, $E^\Lambda_A = \xi E^{(i)}_A + (1 - \xi)E^m$, as well as the Hill upper bound given in fig. 3.6) for the scaling used in fig. 3.6 and are, accordingly, not shown.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig36.png}
\caption{Bounds and estimates for the effective transverse Young’s moduli of glass/epoxy fiber reinforced composites as functions of the fiber volume fraction.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig37.png}
\caption{Bounds and estimates for the effective axial shear moduli of glass/epoxy fiber reinforced composites as functions of the fiber volume fraction.}
\end{figure}
statistical parameters evaluated by Torquato and Lado (1992) for fiber volume fractions $\xi \leq 0.7$. The coefficients of thermal expansion were evaluated on the basis of the relations of Rosen and Hashin (1970).

Generally, a qualitatively similar behavior to the particle reinforced case can be observed. It is noteworthy that the overall transverse CTEs in fig. 3.10 at low fiber volume...
fractions exceeds the CTEs of both constituents. Such behavior is typical for continuously reinforced composites and is caused by the marked axial constraint enforced by the fibers. As expected for continuously reinforced materials, there is little variation between the predictions of the different models for the axial thermal expansion behavior, fig. 3.9. The Mori–Tanaka estimates correspond to the upper bound for the transverse CTE in analogy to the macroscopically homogeneous case. For the axial CTE, however, the Mori–Tanaka results agree with the lower bound, which is a consequence of the axial constraint introduced by the fibers.

![Figure 3.10: Estimates and bounds for the effective transverse CTEs of glass/epoxy fiber reinforced composites as functions of the fiber volume fraction.](image)

In figs. 3.2 to 3.8 the classical self-consistent scheme is not in good agreement with the three-point bounds shown, because the latter explicitly correspond to matrix–inclusion topologies. Considerably better agreement with the CSCS can be obtained by using three-point parameters of the overlapping sphere or cylinder type (which can also describe cases where both phases percolate, but are not as symmetrical with respect to the constituents as the CSCS). From a practical point of view it is worth noting that despite their sophistication improved bounds (and higher order estimates) may give overly optimistic predictions for the overall moduli because they describe ideal composites, whereas in actual “two-phase” materials it is practically impossible to avoid flaws such as porosity.

Before closing this chapter it is worth mentioning that more complex responses may be obtained when at least one of the constituents is transversely isotropic and one of the elastic or conductivity contrasts exceeds unity whereas the other is smaller (such situations can occur, e.g., in metals reinforced by carbon fibers). Also, the differences between the predictions of different mean-field models typically tend to be more pronounced when these algorithms are used to describe nonlinear responses, e.g., elastoplastic behavior. Furthermore, the angular dependences of stiffnesses and CTEs in fiber reinforced materials can be quite rich, compare, e.g., Pettermann et al. (1997).
3.6 Comparisons of Mean-Field and Bounding Predictions for Effective Conductivities

Predictions for the effective conductivities of composites as functions of the phase volume fractions are qualitatively similar to the corresponding data for the elastic moduli, as can be seen in figs. 3.11 and 3.12, which pertain to a polyetherimide matrix reinforced by aligned short graphite fibers of aspect ratio $a = 10$. The plots use material parameters given by Harte and Mc Namara (2006), which are listed in table 3.2. The fibers show transversally isotropic conductivity, the axial conductivity contrast taking a value of approximately 37, whereas the transverse conductivity contrast is an order of magnitude smaller.

Table 3.2: Constituent material parameters of polyetherimide matrix and the T-300 graphite fibers used in generating figs. 3.11 to 3.12.

<table>
<thead>
<tr>
<th></th>
<th>$k_A [\text{Wm}^{-1}\text{K}^{-1}]$</th>
<th>$k_T [\text{Wm}^{-1}\text{K}^{-1}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>matrix</td>
<td>0.22</td>
<td>0.22</td>
</tr>
<tr>
<td>reinforcements</td>
<td>8.40</td>
<td>0.84</td>
</tr>
</tbody>
</table>

Because for the composite described by figs. 3.11 and 3.12 the matrix conductivity is smaller than either the axial or the transverse fiber conductivities, the Mori–Tanaka estimates for the effective conductivity coincide with the lower Hashin–Shtrikman bounds. Even though the selected fiber aspect ratio is rather moderate, the macroscopic conduction

![Figure 3.11](image_url)

Figure 3.11: Bounds and estimates for the effective axial conductivity of composites consisting of aligned short graphite fibers ($a = 10$) in a polyetherimide matrix as functions of the fiber volume fraction.
behavior approaches that of a continuously reinforced composite, with the upper Wiener and Hashin–Shtrikman bounds being nearly identical. As in the elastic case the classical self-consistent scheme, which does not describe matrix–inclusion topologies for all volume fractions, is close to one Hashin–Shtrikman bound at low fiber volume fractions and close to the other at high ones. The predictions of the differential scheme differ rather strongly from the lower Hashin–Shtrikman bounds, a behavior that tends to be especially marked in materials with elevated conductivity contrasts (which may reach very high values, indeed, in the case of electrical conduction).
Chapter 4

General Remarks on Modeling Approaches Based on Discrete Microstructures

In the following, micromechanical approaches based on discrete microstructures are understood to encompass the periodic homogenization, embedding and windowing methods discussed in chapters 5 to 7 and sketched in fig. 1.1. Broadly speaking, these “full field” models trade off potential restrictions to the generality of the phase arrangements against the capability of using fine-grained geometrical descriptions and of resolving details of the stress and strain fields at the length scale of the inhomogeneities\(^{48}\). The most important applications of such methods are homogenizing the behavior of inhomogeneous materials and evaluating the microscopic stress and strain fields in relevant microgeometries at high resolution. The latter information, on the one hand, is important when the local fields fluctuate strongly and, consequently, much information is lost by volume averaging. On the other hand, it is important for understanding the damage and failure behavior of inhomogeneous materials, which in many cases depends on details of their microgeometry.

The most common work flow in models involving discrete microstructures consists of first obtaining appropriate phase arrangements, see section 4.1, discretizing the resulting volume elements and preparing them for application of a numerical engineering method, see sections 4.2 and 4.3, solving for the microfields, and, finally, postprocessing, see section 4.4.

4.1 Microgeometries and Volume Elements

The heterogeneous volume elements (“microgeometries”) used in full field models range from highly idealized periodic geometries (“simple periodic arrays”), such as cubic arrays of spheres in a matrix, to large phase arrangements that aim at closely approximating the geometrical complexity and/or arrangement statistics of actual inhomogeneous materials. There are two philosophies for generating heterogeneous volume elements for modeling, viz., using synthetic microstructures, usually obtained from computer-based simulations,\(^{48}\)

\[^{48}\text{Such methods by construction aim at resolving inter-particle and intra-particle interactions at the maximum level attainable within continuum mechanics.}\]
or employing phase arrangements that are directly based on microgeometries obtained by experimental methods. The present discussion concentrates on microgeometries with matrix–inclusion topology.

**Synthetic Phase Arrangements**

Synthetically generated volume elements that go beyond simple periodic arrays may be classified into two groups, the first of which makes use of volume elements that contain generic arrangements of a number of randomly positioned and, where appropriate, randomly oriented reinforcements. The second group aims at constructing phase arrangements that have phase distribution statistics identical to those of the target material. A further possible approach to synthesizing realistic volume elements by computer algorithms, which consists of modeling relevant production processes, has seen little use to date for composites due to the complexity of the task. For a recent review on generating three-dimensional inhomogeneous volume elements see Bargmann et al. (2018).

Synthesizing generic “multi-inhomogeneity” model geometries with random arrangements of reinforcements in many cases has involved random sequential addition methods\(^{49}\). The reinforcement volume fractions that can be reached by RSA methods tend to be moderate due to jamming (“geometrical frustration”), however, their representativeness for actual phase arrangements generated by mixing processes is open to question (Stroeven et al., 2004), and they may give rise to biased phase arrangements (Pontefisso et al., 2016). Major improvements can be obtained by using RSA geometries, or, in the case of very high target inhomogeneity volume fractions, packings obtained by methods such as collective rearrangement models, see, e.g., Maggi et al. (2008), as starting configurations for random perturbation or “hard core shaking” models, compare, e.g., Buryachenko et al. (2003). The latter algorithms apply small random displacements to each inhomogeneity to obtain improved and, if the size of the volume element is reduced in the process, more tightly packed arrangements of reinforcements. Alternatively, RSA methods may be combined with stirring models (Zangenberg and Brøndsted, 2013). Approaches of the above types have been used to generate (approximately) statistically homogeneous distributions of particulate or fibrous reinforcements, see, e.g., Gusev (1997), Böhm and Han (2001), Lusti et al. (2002), Duschlbauer et al. (2006) or Rasool and Böhm (2012), as well as idealized clustered microgeometries (Segurado et al., 2003). Alternatively, random sequential addition may be combined with optimization procedures (Pathan et al., 2017). RSA-based methods can also be adapted for obtaining volume elements that follow prescribed fiber orientation distributions, see, e.g., Schneider (2017).

\(^{49}\)In random sequential addition (RSA) algorithms (also known as random sequential insertion or random sequential adsorption models, and sometimes referred to as “static simulations”), positions and, where applicable, orientations for new reinforcements are created by random processes, a candidate inhomogeneity being accepted if it does not “collide” with any of the existing ones and rejected otherwise. In contrast to random packing, in generating microgeometrical models a “collision” often involves the violation of some minimum distance rather than actual touching of or overlapping with neighboring inhomogeneities. Whereas collision checking is straightforward with circular or spherical inhomogeneities, algorithms such as the separating axis method (Schneider and Eberly, 2003) may be required for more complex shapes. If periodic phase arrangements are generated, up to 9 periodic copies must be maintained for any inhomogeneity that is intersected by one or more faces of the volume element.
Matrix–inclusion configurations of elevated inhomogeneity volume fraction can be obtained by “random particle dropping” (Chiaia et al., 1997) or gravitational (Khalevitsky and Konovalov, 2019) methods, discrete-element-based models (Ismail et al., 2016), or molecular-dynamics-like algorithms (Lubachevsky et al., 1991), known as collective rearrangement models, compare, e.g., Ghossein and Lévesque (2013). Furthermore, arrangements of particles in composites may be generated via hydrodynamic interactions in a pseudo-fluid (Elliott and Windle, 2000). For a discussion of methods for generating geometrically complex volume elements containing fibrous inhomogeneities that are compliant in bending, see, e.g., Altendorf and Jeulin (2011).

Additional approaches to generating multi-inhomogeneity synthetic volume elements with matrix–inclusion topology have been based on “eroding” (or shrinking) the individual cells of suitable Voronoi tesselations (Fritzen and Böhlke, 2011) of the whole volume, so that each cell gives rise to a polyhedral particle, or on first filling the unit cell with tetrahedra which are then assigned to the required phases by a suitable algorithm (Galli et al., 2008), randomness being introduced by the underlying tesselation in both cases. Both methods result in arrangements of randomly positioned and oriented inhomogeneities of irregular polygonal or polyhedral shape, the first mentioned one being applicable to a wide range of inhomogeneity volume fractions.

The most important alternative strategy for generating volume elements by simulation aims at obtaining “statistically reconstructed” phase arrangements, which are not identical to any given sample, but rather are statistically equivalent to the material to be modeled. Statistical reconstruction typically gives rise to optimization problems in which some starting configuration (e.g., a periodic phase arrangement of appropriate volume fraction) is adapted such that suitable statistical and/or stereological descriptors of the phase distributions approach the chosen target descriptor(s) as closely as possible. Procedures for reconstructing matrix–inclusion and more general microgeometries have been reported that employ simulated annealing procedures (Rintoul and Torquato, 1997; Torquato, 1998b; Bochenek and Pyrz, 2004), genetic algorithms (Zeman and Šejnoha, 2001), and other minimization methods (Roberts and Garboczi, 1999). For in-depth discussions of the underlying issues, such as statistical descriptors for the phase arrangements of inhomogeneous materials, see, e.g., Torquato (2002) or Zeman (2003); questions of the uniqueness of reconstructed two-phase microgeometries were discussed by Jiao et al. (2007). Alternative approaches to statistical reconstruction were reported, e.g., by Vaughan and McCarthy (2010) or de Francqueville et al. (2019). Studies on statistical reconstruction have been based on a number of statistical descriptors of phase arrangements, among them n-point, nearest neighbor and radial distribution functions as well as correlation functions, i.e., they follow the concept of geometrical RVEs discussed in section 1.3. Microstructures generated by statistical reconstruction are especially attractive because the phase arrangements in at least some actual composites were found to be not completely spatially random (Trias, 2005). For volume elements that contain considerable numbers of inhomogeneities or other microstructural features, statistically reconstructed phase arrangements are more specific to a given target material than are generic random microgeometries. Because at present computational requirements limit the sizes of volume elements that can be handled routinely by numerical engineering methods, this advantage has been of limited practical
impact in the context of continuum micromechanics. This issue can be addressed by the concept of “statistically similar volume elements”, in which SVEs are optimized to approach statistical descriptors of the target material as closely as possible, see, e.g., Balzani et al. (2010) and Scheunemann et al. (2015).

Synthetically generated matrix–inhomogeneity microgeometries have tended to employ idealized reinforcement shapes, equiaxed particles embedded in a matrix, for example, being often represented by spheres, and fibers by cylinders or prolate spheroids of appropriate aspect ratio. However, recent work has also been aimed at polyhedral inhomogeneities, see, e.g., Nogales and Böhm (2008), Rasool and Böhm (2012), Zhang et al. (2014) or Böhm and Rasool (2016).

Real Structure Phase Arrangements

Instead of generating phase arrangements by computer algorithms, volume elements may be chosen to follow as closely as possible the actual microgeometry in some appropriate subvolume of the material to be modeled, obtained from metallographic sections (Fischmeister and Karlsson, 1977), serial sections (Terada and Kikuchi, 1996; Li et al., 1999), tomographic data (Hollister et al., 1994; Kenesei et al., 2004; Chawla and Chawla, 2006; Buffière et al., 2008), etc. The resulting volume elements are often called “real microstructure” models.

The generation of real structure models from pixel (digital images) or voxel (tomographic) data describing the geometries of inhomogeneous materials has been the focus of considerable research efforts. Such work involves the steps of selecting from experimental data sets appropriate volume elements for analysis ("registration") and of identifying the regions occupied by the different constituents by thresholding of the grey values of the pixels or voxels ("segmentation"). At this stage pixel or voxel models (compare section 4.3) can be generated directly from the segmented data set or contouring procedures may be used for obtaining “smooth” phase domains, the latter operation typically being more manpower intensive. Alternatively, irregular particle shapes in real structure arrangements have been approximated by ellipsoids of appropriate size, shape and orientation, compare Li et al. (1999). Computed tomography has proven especially useful for determining the microgeometries of inhomogeneous materials with constituents that differ considerably in X-ray absorption, e.g., porous and cellular materials.

Real microstructure models provide accurate descriptions of actual phase arrangements, which, however, may depend to a considerable extent on details of the underlying experiments, e.g., the resolution of the digital images, as well as of the registration, segmentation and, where applicable, smoothing procedures. In general, the resulting volume elements

---

50 For uniform boundary conditions it can be shown that the overall elastic behavior of matrix–inclusion type composites can be bounded by approximating the actual shape of particles by inner and outer envelopes of “smooth” shape, e.g., inscribed and circumscribed ellipsoids. This is known as the Hill modification theorem (Hill’s comparison theorem, auxiliary theorem or strengthening theorem), compare Hill (1963) and Huet et al. (1990). Approximations of actual inhomogeneity shapes by ellipsoids typically work considerably better for convex than for non-convex particle shapes (Kachanov and Sevostianov, 2005).

51 For a review or work directed at automatically generating high quality structured meshes from voxel data sets, see, e.g., Young et al. (2008).
obviously are non-periodic, which restricts their use with the periodic homogenization tech-
niques discussed in chapter 5, but makes them well suited to windowing and embedding
approaches, compare chapters 6 and 7.

Sizes of Volume Elements

When small volume elements are used in full field simulations, the predicted macroscopic
responses tend to show a marked dependence on the size of the volume elements, see, e.g.,
Iorga et al. (2008). This immediately raises the question of the size of volume element
required for adequately capturing the macroscopic physical behavior of the material to be
studied\(^{52}\). On the one hand, the requirement of limiting numerical cost obviously puts a
premium on using the smallest viable volume element but, on the other hand, resolving
microgeometry effects militates for using microgeometries that are or approximate repre-
sentative volume elements (RVEs, compare section 1.3). These conflicting demands led to
the concepts of “minimum RVEs” by Ren and Zheng (2004), of “statistical RVEs” by Trias
et al. (2006) and of statistical volume elements by Ostoja-Starzewski (2006).

When the philosophy of geometrical RVEs is followed, estimates for suitable sizes of
volume element can be obtained on the basis of descriptors of the microgeometry alone,
the physical property to be modeled playing no role. In such a context the adequacy of
the size of a volume element may be assessed, e.g., on the basis of experimentally obtained
correlation lengths (Bulsara et al., 1999) or covariances (Jeulin, 2001) of the phase ar-
rangement or by posing the requirement of having at least two statistically independent
inhomogeneities in the volume element (Zeman and Šejnoha, 2001). Such approaches can
be extended to anisotropic microgeometries (Wang et al., 2019) and they have proven suc-
cessful for micromechanical models of the elastic behavior of inhomogeneous materials.

The concept of physical RVEs as stated in section 1.3, in contrast, implies that the
suitability of a given size of volume element for micromechanical modeling depends on
the physical property to be studied. Assessing the representativeness of a volume element
in this sense can, in principle, be carried out by windowing analysis, compare chapter 6,
with identical predictions of the macroscopic behavior under macroscopically homogeneous
stress and strain boundary conditions, respectively, being indicative of physical RVEs in
the strict sense. In practice, this criterion has proved to be difficult to fulfill in a rigorous
way, and, consequently, the suitability of volume elements for a given task is typically
assessed by specifying appropriate thresholds or by checking some necessary criteria for
representativeness. The most important approach of this type is based on studying the
convergence of estimates for the macroscopic behavior with growing size of the volume el-
ments, compare, e.g., Savvas et al. (2016). Furthermore, the compliance of the predicted
effective responses with tight bounds (e.g., three-point bounds in the case of randomly
positioned spheres of moderate phase contrast) and the attaining of appropriate macro-
scopical material symmetry may be checked for. For comparisons of different geometry and
microfield based criteria for choosing the size of model geometries see, e.g., Trias et al.
(2006) and Moussady et al. (2013). It must be kept in mind, though, that fulfilling the

\(^{52}\)Traditionally, RVEs have been defined in terms of macroscopic responses. Assessing higher statis-
tical moments of phase-level microstresses in addition to the first one (which underlies modulus-based
considerations) appears feasible but may lead to more stringent requirements on RVEs.
latter criteria — while going a considerable way towards ensuring useful modeling results — is not sufficient for establishing proper RVEs.

Assessments based solely on the geometry or on the overall elastic behavior predict that relatively small volume elements can give fairly accurate results. Zeman (2003) reported that the transverse elastic behavior of composites reinforced by continuous fibers can be satisfactorily described by unit cells containing reconstructed arrangements of 10 to 20 fibers. “Pragmatic” definitions, in which a given volume element must fulfill some given criterion to a specified accuracy to be accepted as a physical RVE, have been used to a considerable extent. Using a nonlocal Hashin–Shtrikman model Drugan and Willis (1996) found that, for statistically isotropic composites consisting of a matrix reinforced by spherical particles, volume elements with sizes of some two and five particle diameters are sufficient for obtaining errors of less than 5% and less than 1%, respectively, in terms of the macroscopic elastic stiffness. Interestingly, these sizes of volume elements come out as being independent of the particle volume fraction, a prediction that is not shared by alternative models for assessing RVE sizes (Pensée and He, 2007; Xu and Chen, 2009). Considerably larger volume elements are required for obtaining a given level of accuracy for aligned ellipsoidal inhomogeneities, where the RVE size depends on the reinforcement volume fraction (Monetto and Drugan, 2009), and different estimates for the RVE size result when modeling elastic and thermal conduction behavior (Kanit et al., 2003).

For cases involving inelastic constituent behavior, a number of numerical studies (Zohdi, 1999; Jiang et al., 2001; Böhm and Han, 2001) have indicated that larger volume elements tend to be required for satisfactorily approximating the overall symmetries and for obtaining good agreement between the responses of different phase arrangements designed to be statistically equivalent. More recent studies (Cugnoni and Galli, 2010; Galli et al., 2012; Zhang et al., 2014) have confirmed that relatively large volume elements are required for approaching representativeness in elastoplastic composites, especially at elevated strains. They report a clear dependence of the RVE size on the inhomogeneity volume fraction and on the macroscopic strain for particle reinforced ductile matrix composites. The main reason for this behavior lies in the marked inhomogeneity of the microscopic strain fields that is typically present in nonlinear composites. For example, there tend to be contiguous zones of concentrated plastic strains, evolve to become considerably larger than individual reinforcements, thus effectively introducing a new length scale into the problem. In the case of path dependent material behaviors assessments of the suitability of a given volume element, strictly speaking, pertain only to the load paths actually considered. Eliminating the dependence of the homogenized response on the size of the volume element can be

\[ \xi \]

For a given level of uncertainty, this corresponds to the number of identical inhomogeneities within the volume element scaling with their volume fraction, \( \xi \).

In the hardening regime, phase arrangements with high reinforcement volume fractions may give rise to smaller RVEs than do volume elements with moderate ones, the formation of zones of concentrating plastic strains being impeded by elastic inhomogeneities (Cugnoni and Galli, 2010). For elastoplastic matrices, weaker strain hardening typically produces more inhomogeneous microstrains, which leads to a requirement for bigger volume elements that contain a higher number of inhomogeneities. Extremely large volumes were reported to be necessary when one of the phases shows softening, e.g., due to damage (Zohdi and Wriggers, 2001; Swaminathan and Ghosh, 2006; Gitman, 2006). For elastic polycrystals, the anisotropy and shape of the grains were also reported to influence the required size of volume elements (Ren and Zheng, 2004).
argued, in fact, to become impossible in the presence of strain localization (Gitman et al., 2007), the statistical homogeneity of the whole sample or component being lost. Large volume elements are also required for materials the behavior of which depends strongly on “rare features” of the microgeometry and for configurations that contain features covering a range of length scales, e.g., randomly oriented fibers of elevated aspect ratio. Comparisons between equivalent volume element sizes for various macroscopic behaviors of two-dimensional inhomogeneous materials were reported by Ostoja-Starzewski et al. (2007).

Modeling work in most cases has been based on volume elements that are known to be of insufficient size to be proper RVEs, the main reasons for using them being limits in the size of models that can be handled and difficulties in providing suitable RVEs for actual materials. Such “sub-RVE” volume elements, which may be periodic or non-periodic, are often referred to as statistical volume elements (SVEs, Ostoja-Starzewski (2006)), testing volume elements (TVEs, Diebels et al. (2005)), surrogate volume elements, or windows, compare chapter 6 — calling them RVEs is not good practice. When a number of SVEs of comparable volume and pertaining to a given inhomogeneous material are available, they may be viewed as being different realizations of the phase arrangement statistics describing that material. In such cases ensemble averaging over the results obtained from sets of volume elements can be used to obtain improved estimates for the effective material properties, compare, e.g., Kanit et al. (2003) and Stroeven et al. (2004). The number of different volume elements required for a given level of accuracy of the ensemble averages decreases as their size increases (Khisaeva and Ostoja-Starzewski, 2006). Furthermore, for macroscopically isotropic materials the anisotropic contributions to the ensemble averaged results was reported to be markedly reduced compared to the predictions of the individual SVEs (El Houdaigui et al., 2007). It was, however, reported by Galli et al. (2012) that ensemble averaging over non-periodic statistical volume elements in the elastoplastic range may be compromised by boundary layers.

For linear properties Kanit et al. (2003) proposed confidence intervals for assessing the quality of the ensemble averaged results obtained from sets of SVEs. In this context an error measure can be defined in terms of the standard deviation $S(Y)$ of some given modulus or tensor element, $Y$, as

$$\text{err}(Y) = 1.96 \frac{S(Y)}{\sqrt{n_{\text{SVE}}}} , \quad (4.1)$$

where $n_{\text{SVE}}$ is the number of statistically equivalent SVEs used in ensemble averaging. A related approach for nonlinear behavior was proposed by Pelissou et al. (2009). A further refinement consists in weighting the contributions of the SVEs according to statistical parameters (Qidwai et al., 2012).

The size of the volume element(s) used in modeling affects not only the macroscopic responses, but also the predicted microfields. Accordingly, some authors have explored the question of what size of volume element is required for studying the local fields at some specific location. This may be done by studying series of volume elements of different size that pertain to a given microstructure, see, e.g., Ozturk et al. (2016).
4.2 Boundary Conditions

As the size of volume elements describing inhomogeneous materials is increased, the resulting predictions of the apparent overall behavior improve until representativeness is achieved, when further increases in model size do not lead to further changes in the results and the effective behavior is obtained. The rate of convergence towards proper effective behavior depends on the boundary conditions applied to the volume element.

Fulfilling the surface integral version of Hill’s macrohomogeneity condition, eqn. (1.7), which can be written in the form

\[ \int_\Gamma [\mathbf{t}(\mathbf{x}) - \langle \mathbf{\sigma} \rangle \star \mathbf{n}_\Gamma(\mathbf{x})]^T [\mathbf{u}(\mathbf{x}) - \langle \mathbf{\epsilon} \rangle \star \mathbf{x}] d\Gamma = 0 , \]  

see Hill (1967) and Hazanov (1998), for inhomogeneous volume elements of finite size, can be achieved by four types of boundary conditions, three of which are based on uniform B.C.s (Hazanov and Amieur, 1995; Ostoja-Starzewski, 2006).

First, the traction term in eqn. (4.2) can be made to vanish over the whole boundary by specifying appropriate Neumann boundary conditions for the tractions \( \mathbf{t}(\mathbf{x}) \). These are obtained by prescribing a macroscopically homogeneous stress tensor \( \mathbf{\sigma}^a \) on all faces of the volume element,

\[ \mathbf{t}(\mathbf{x}) = \mathbf{\sigma}^a \star \mathbf{n}_\Gamma(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma , \quad (4.3) \]

leading to statically uniform boundary conditions (SUBC, or uniform Neumann BC, UNBC).

Second, the right hand term in eqn. (4.2) can be enforced to be zero by imposing a given macroscopically homogeneous strain tensor \( \mathbf{\epsilon}^a \) on all boundary surfaces,

\[ \mathbf{u}(\mathbf{x}) = \mathbf{\epsilon}^a \star \mathbf{x} \quad \forall \mathbf{x} \in \Gamma , \quad (4.4) \]

resulting in kinematically uniform boundary conditions (KUBC, or uniform Dirichlet BC, UDBC). Because eqns. (4.3) and (4.4) impose homogeneous stress or strain fields on the boundary of the volume element, they are known as macrohomogeneous boundary conditions. Some authors also refer to them as Hashin boundary conditions.

Third, mixed uniform boundary conditions (MUBC) may be specified, in which the scalar product under the integral is made to vanish separately for each face \( \Gamma_k \) that is part of the surface of the volume element,

\[ [\mathbf{t}(\mathbf{x}) - \langle \mathbf{\sigma} \rangle \star \mathbf{n}_\Gamma(\mathbf{x})]^T [\mathbf{u}(\mathbf{x}) - \langle \mathbf{\epsilon} \rangle \star \mathbf{x}] d\Gamma = 0 \quad \forall \mathbf{x} \in \Gamma_k . \quad (4.5) \]

This involves appropriate combinations of traction and strain components that are uniform over a given face of the volume element rather than specifying a macroscopically homogeneous field. Mixed uniform boundary conditions that fulfill eqns. (4.2) and (4.5) must be orthogonal in their fluctuating contributions (Hazanov and Amieur, 1995). The

---

Equation (4.2) essentially states that energy equivalence is achieved when local fluctuations do not contribute to the elastic strain energy density.
symmetry boundary conditions discussed in section 5.2 are a type of MUBC and specific sets of MUBC useful for window-type microgeometries are presented in chapter 6.

Finally, the displacement and traction fields may be decomposed into slow and fast contributions in analogy to eqn. (1.2),

$$\mathbf{u}(\mathbf{x}) = \langle \mathbf{\varepsilon} \rangle * \mathbf{x} + \mathbf{u}'(\mathbf{x}) \quad \text{and} \quad \mathbf{t}(\mathbf{x}) = \langle \mathbf{\sigma} \rangle * \mathbf{n}_{\Gamma}(\mathbf{x}) + \mathbf{t}'(\mathbf{x})$$ \hspace{1cm} (4.6)

Inserting these expressions into the Hill–Mandel criterion, eqn. (4.2), directly leads to the condition

$$\int_{\Gamma} \mathbf{t}'(\mathbf{x})^T \mathbf{u}'(\mathbf{x}) d\Gamma = 0 \hspace{1cm} (4.7)$$

which are fulfilled by periodic phase arrangements. By definition, the latter can be fully described by a single periodic volume element, called a unit cell, the surface of which must consist of pairs of parallel surface elements, compare section 5.2. For homologous points on such pairs of faces the stress and strain tensors must be identical, which implies that the displacement fluctuation vectors $\mathbf{u}'(\mathbf{x})$ at the two faces are identical, too, whereas the traction fluctuation vectors $\mathbf{t}'(\mathbf{x})$ have equal absolute values but opposite orientations, compare fig. 4.1. As a consequence, contributions to eqn. (4.7) from each pair of homologous faces cancel out and the Hill condition is fulfilled for the periodic unit cell itself and for larger periodic assemblages of unit cells. A more detailed discussion of periodic unit cells and the associated periodicity B.C. is given in section 5.2. Incidentally, if the shape of a volume element is such that for each surface point there exists an “antipodic” point having the opposite normal vector (as is the case, e.g., for spheres), eqn. (4.7) can be formally enforced in a point-wise way even for non-periodic volume elements (Glüge et al., 2012), giving rise to so-called “antipodic periodicity” boundary conditions.

![Figure 4.1: Sketch of antiperiodic traction fluctuations at faces of periodic VE under macroscopic uniaxial tensile loading.](image)

Macrohomogeneous boundary conditions following eqns. (4.3) and (4.4) can be shown to give rise to lower and upper estimates, respectively, for the overall elastic tensor of a given mesoscopic volume element (Nemat-Nasser and Hori, 1993). Results obtained with mixed uniform and periodicity boundary conditions lie between the corresponding lower and upper estimates (Hazanov and Huet, 1994) and show a faster convergence towards the effective response with increasing volume element size, periodicity B.C. being best in terms of convergence.
SUBC and KUBC do not pose restrictions on the phase geometry or on the shape of volume elements, but some MUBC are subject to certain restrictions on the latter count. These boundary conditions form the basis of windowing methods, see chapter 6. Periodic homogenization obviously requires specific volume elements describing periodic phase geometries (which restricts admissible shapes), to which periodicity boundary conditions or, in special cases, symmetry and antisymmetry boundary conditions are applied, compare section 5. Antipodic boundary conditions require volume elements of specific shapes, but are not restricted to periodic geometries.

In addition to the above four types of boundary condition, the Mandel–Hill criterion can also be fulfilled by configurations in which the volume element proper is surrounded by a layer of homogeneous material that is given the effective behavior of the core. Elastic tensors obtained by applying macrohomogeneous or periodicity boundary conditions, eqns. (4.3) to (4.7), to the outer surface of the embedding layer can be shown to fall within the upper and lower estimates generated by applying the macrohomogeneous B.C. directly to the volume element, see Temizer et al. (2013). Such configurations form the basis of self-consistent embedding schemes, compare chapter 7.

Among the above types of models and boundary conditions only fully implemented periodicity B.C. do not lead to boundary layers in the predicted microscopic stress and strain fields\textsuperscript{56}, which is an advantage since boundary perturbations may lead to spurious behavior when nonlinear constitutive response of the constituents is considered. Periodic homogenization also is unique in that volume elements consisting of multiple copies of a given unit cell give rise to the same prediction as the unit cell itself (provided wave phenomena or buckling are not involved).

4.3 Numerical Engineering Methods

The majority of continuum micromechanical studies of discrete microstructures have employed standard numerical engineering methods for resolving the microfields. Work reported in the literature has involved Finite Difference (FD) and Finite Volume algorithms, compare Adams and Doner (1967), Bansal and Pindera (2006) or Khatam and Pindera (2009), spring lattice models, compare Ostoja-Starzewski (2002), the Boundary Element Method (BEM), compare Achenbach and Zhu (1989) or Liu et al. (2005b), as well as the Finite Element Method (FEM) and its developments such as Extended Finite Element Methods, mesh-free and particle methods, compare Sukumar et al. (2001), Dang and Sankar (2007) and Missoum-Benziane et al. (2007), or FE-based discrete dislocation models (Cleveringa et al., 1997). In general, spring lattice models tend to show some advantages in handling pure traction boundary conditions and in modeling the progress of microcracks due to local (brittle) failure. Boundary element methods typically are at their best in studying geometrically complex linear elastic problems.

\textsuperscript{56}Perturbations caused by the application of uniform boundary conditions to inhomogeneous surfaces or by the local incompatibility between homogenized and inhomogeneous regions depend on the local material contrasts and do not necessarily become smaller with increasing volume element size.
In addition to the above methods, techniques using Fast Fourier Transforms (FFT), compare Moulinec and Suquet (1994), and Discrete Fourier Transforms (DFT), compare Müller (1996), have found considerable use in continuum micromechanics. Typically, the starting point for such iterative algorithms is a Lippmann–Schwinger equation, compare eqn. (2.32). The convolution integral in this equation is solved in Fourier space (where it reduces to a tensor contraction) and the strains are updated in physical space within each iteration, transforms and back transforms between the two spaces being handled by fast numerical algorithms. Such spectral methods are directly applicable to analyzing periodic volume elements, where they tend to be highly efficient, compare, e.g., Michel et al. (1999) or Ghossein and Lévesque (2012), for both linear and nonlinear phase behavior. Further developments of these approaches have allowed the handling of infinite phase contrasts (Brisard and Dormieux, 2010). Homogenization via FFT at present is a highly active research field and FFT-based schemes are becoming the methods of choice for handling large, periodic, voxel-type models. DFT methods have found use in studying the evolution of microstructures (Dreyer et al., 1999).

A further approach, the Transformation Field Analysis of Dvorak (1992), allows the prediction of the nonlinear responses of inhomogeneous materials based on either mean-field descriptions (compare the remarks in section 2.5) or on full field approximations. High computational efficiency is claimed for “classical” TFA models employing piecewise uniform transformation fields (Dvorak et al., 1994), and the extension to FE²-type models has been reported (Marfia and Sacco, 2018). A development of the TFA, the Nonuniform Transformation Field Analysis (Michel and Suquet, 2004), expands the inelastic strains into a number of nonuniform, incompressible and orthogonal “plastic flow modes” to achieve efficient descriptions of the elastoplastic behavior of inhomogeneous materials. A number of additional specialized approaches are discussed in connection with periodic microfield analysis, see section 5.1.

Obviously, when numerical engineering methods are used in continuum micromechanics, the characteristic length of the discretization (“mesh size”) must be chosen considerably smaller than the microscale of the considered problem in order to obtain spatially well resolved results.

At present, the FEM is the most popular numerical scheme for evaluating full field models, especially in the nonlinear range, where its flexibility and capability of supporting a wide range of constitutive descriptions for the constituents and the interfaces between them are especially appreciated. An additional asset of the FEM in the context of continuum micromechanics is its ability to handle discontinuities in the stress and strain components (which typically occur at interfaces between different constituents) in a natural way via appropriately placed element boundaries. Furthermore, Finite Element codes are widely available and many of them provide linear constraint equations for coupling multiple degrees of freedom, which are required for periodic homogenization, compare section 5.2.

\footnote{Constitutive models for constituents used in FEM-based micromechanics have included a wide range of elastoplastic, viscoelastic, viscoelastoplastic and continuum damage mechanics descriptions as well as crystal plasticity models, see, e.g., McHugh et al. (1993), and nonlocal models, compare Bassani et al. (2001) or Drabek and Böhm (2006). In addition, the FEM has supported a range of modeling options for interfaces between phases.}
Applications of (more or less) standard Finite Element methods to micromechanical studies may be classed into five main groups, compare fig. 4.2, all of which involve specific trade-offs in terms of complexity of the models, required meshing effort, computational efficiency and aspects of accuracy.

In most published work in FE-based micromechanics the phase arrangements are discretized by an often high number of “standard” continuum elements, the mesh being designed in such a way that element boundaries (and, where appropriate, special interface elements) are positioned at all interfaces between constituents. Such “classical meshing approaches” typically use unstructured meshes and have the advantage that in principle any microgeometry can be discretized at the required level of accuracy, making them the most flexible way of using the FEM in continuum micromechanics. The resulting meshes are suitable for analysis with many readily available FE packages. However, the actual modeling of complex phase configurations in many cases requires sophisticated and/or specialized preprocessors for generating the mesh, a task that may be work intensive and has been found to be difficult to automatize\textsuperscript{58}, especially in the case of periodic phase arrangements. In addition, the resulting stiffness matrices may show unfavorable conditioning due to suboptimal element shapes. Classical meshing approaches are capable of highly resolving the microfields at local “hot spots” (e.g., between closely neighboring reinforcements), but the resulting mesh refinements can lead to very large models, indeed. Nevertheless, its built-in capabilities for providing mesh refinement where it is needed and for using standard mesh refinement procedures are major strengths of this discretization strategy.

Alternatively, a smaller number of specialized hybrid finite elements may be used, which are specifically formulated to model the deformation, stress, and strain fields in an inhomogeneous region consisting of a single inhomogeneity or void together with the surrounding matrix on the basis of some appropriate analytical theory (Accorsi, 1988). The most highly developed approach of this type at present is the Voronoi Finite Element Method (Ghosh et al., 1996), in which the mesh for the hybrid elements is obtained from Voronoi tessellations based on the positions of the reinforcements. This way the meshing effort can be kept rather low and large planar multi-inhomogeneity arrangements can be analyzed using a limited number of (albeit rather complex) elements. Numerous phase-level constitutive and damage models have been implemented into the method and an extension to three dimensions was reported (Ghosh and Moorthy, 2004). Modeling approaches based on hybrid elements are specifically tailored to inhomogeneous materials with matrix–inclusion topologies, and good accuracy as well as significant gains in efficiency have been claimed for them.

A computational strategy that occupies a position between the above two approaches uses static condensation to remove the degrees of freedom of the interior nodes of the inho-

\textsuperscript{58} Most preprocessors for Finite Element analysis are not geared towards discretizing matrix–inclusion topologies with thin matrix bridges between closely neighboring inhomogeneities (finely resolving the microfields in such regions has limited influence on the elastic and small-strain plastic macroscopic responses, but tends to be important in evaluating the microfields and may be critical for modeling damage initiation). Other major sources of practical difficulties are intersections between phase and cell boundaries at very acute or obtuse angles and the generation of the periodic meshes at the surfaces of volume elements required for PMAs.
mogeneities (or, in the case of coated reinforcements, of inhomogeneities plus interphases) from the stiffness matrix of a heterogeneous volume element (Liu et al., 2005a), the matrix region being meshed by standard elements. Such methods may considerably reduce the computational requirements of linear multi-inhomogeneity models.

Especially when the phase arrangements to be studied are based on digital images of actual microgeometries, a third approach to discretizing microgeometries is of interest. It involves using a mesh consisting of regular, rectangular or hexahedral, elements of fixed size having the same resolution as the digital data, each element being assigned to one of the constituents by operations such as thresholding of the grey values of the corresponding pixel or voxel. The use of structured meshes in such “digital image based” (DIB) models has the advantage of allowing a straightforward automatic model generation from appropriate experimental data (metallographic sections, tomographic scans) and of avoiding ambiguities in smoothing the digital data (which are generally present if “standard”, unstructured FE meshes are employed for discretizing pixel- or voxel-based data of this type). However, modeling approaches that directly discretize pixel or voxel geometries obviously lead to ragged phase boundaries, which may give rise to some oscillatory behavior of the solutions (Niebur et al., 1999), can cause high local stress maxima (Terada et al., 1997), may degrade accuracy, and negate detailed modeling of interfacial effects. Because the resolution of the microgeometry is determined by the chosen voxel size, mesh size control tends to be a difficult issue with DIB models, with global mesh refinement by supersampling tending to be an inefficient approach. Some of these limitations can, however, be addressed by local smoothing algorithms, compare Boyd and Müller (2006). Despite the above concerns, pixel- or voxel-based models have been claimed not to cause unacceptably large errors in the predicted macroscopic behavior even for relatively coarse discretizations, at least in the linear elastic range (Guldberg et al., 1998), where they have found wide use. In nonlinear regimes, however, local stress concentrations at ragged interfaces may give rise to considerable errors which can be attenuated by averaging procedures, compare Fang et al. (2016).
When synthetic volume elements with matrix–inclusion topology are studied by voxel-based methods, care may be required to ensure that closely neighboring inhomogeneities are properly resolved by the structured mesh and that their spurious “merging” is avoided. Most limitations of voxel-based Finite Element models also tend to apply to FFT-based micromechanical methods, which also make use of highly regular, structured, spatial discretizations.

A fourth, related approach also uses structured FE meshes, but assigns phase properties at the integration point level of standard elements (“multi-phase elements”), see, e.g., Schmauder et al. (1996) or Quilici and Cailletaud (1999). Essentially, this amounts to trading off the ragged boundaries caused by the voxel mesh against smeared-out (and typically degraded) microfields within any element that contains a phase boundary, standard FE shape functions being of limited suitability for handling stress or strain discontinuities within elements. With respect to the element stiffnesses the latter concern can be countered by overintegrating elements containing phase boundaries, which leads to improved approximations of integrals involving non-smooth displacements by numerical quadrature, see Zohdi and Wriggers (2001). The resulting stress and strain distributions, however, remain smeared-out approximations in elements that contain phase boundaries. Multi-phase element models are, accordingly, also limited in their capabilities for detailed modeling of interfaces. An alternative technique giving rise to a similar pattern of strengths and limitations consists in prescribing suitably modified material properties to all elements containing a phase boundary (Toulemonde et al., 2008). In a recent development, “composite voxel” techniques (Kabel et al., 2017; Mareau and Robert, 2017) have been introduced within the framework of FFT-based homogenization, stiffnesses in both linear and nonlinear regimes being approximated in a consistent way, e.g., via laminate models.

The idea of reducing the considerable effort implied in classical meshing approaches has led to applying a number additional of Finite Element modeling techniques to continuum micromechanics that make use of structured “base meshes” which are suitably modified for handling inhomogeneous phase arrangements. Strategies of this type have been based on the Extended Finite Element method (XFEM) or the Generalized Finite Element Method (GFEM), phase boundaries that pass through individual elements being handled via appropriately enriched, non-smooth shape functions, see, e.g., Moës et al. (2003), Legrain et al. (2011), Soghrati et al. (2012) and Dunant et al. (2013). The resulting models can closely follow the shapes of phase boundaries, compare the sketch in fig. 4.2e, and are subject to few restrictions in terms of microgeometries that can be handled.

In addition, there are some FE-based micromechanical methods that do not fall within the groups shown in fig. 4.2. Among them are “embedded element” (embedded reinforcement) or “domain superposition” techniques, which aim at achieving considerable simplifications by avoiding conformal meshing of individual reinforcements. Such approaches use a typically structured and often relatively coarse mesh of continuum elements (“host mesh”) for the matrix. This is combined with independently meshed beam elements, see, e.g., Harper et al. (2012), solid elements, compare, e.g., Jiang et al. (2008), or shell ele-

---

59 It is not clear, however, to what extent structured and voxel-type meshes introduce a systematic bias into the orientation of regions of elevated strain in highly loaded, nonlinear regimes.
ments, see, e.g., Matveeva et al. (2014), that represent the inhomogeneities. The latter are superposed over the host mesh, the different parts of the model being tied together, e.g., by constraint equations. Such strategies imply the presence of excess volume in the models (“volume redundancy”), giving rise to a tendency towards overestimating the overall stiffness, especially at non-dilute volume fractions, which, however, can be compensated for in some cases, compare Hoffmann (2012). Domain superposition approaches markedly reduce the meshing effort in FE-based micromechanical analysis, but tie all displacement fields to the shape functions specified for the host mesh, which, in turn, tends to constrain the local stress and strain fields attainable in either constituent. An alternative FE-based approach to studying composites reinforced by long fibers that are neither straight nor aligned is based on meshing the latter by “strings” of single volume elements and filling the matrix space in-between by a suitable volume mesh (Fliegener et al., 2014). Again, local resolution of the microfields is traded off against the capability of handling complex fiber reinforcement geometries.

A relatively recent development for studying microgeometries involving large numbers of inhomogeneities (hundreds to thousands) by Finite Element methods involves finite element programs specially geared towards solving micromechanical problems. Such codes may make use, e.g., of matrix-free iterative solvers such as Conjugate Gradient (CG) methods, analytical solutions for the microfields (such as constant strain approximations corresponding to the upper Hill bounds, eqn. (3.1), being employed as starting solutions to speed up convergence\(^6\)). For micromechanical studies involving such solvers see, e.g., Gusev (1997), Zohdi and Wriggers (2001) or Arbenz et al. (2008). An alternative approach in this context are multi-grid solvers, compare, e.g., Gu et al. (2016). Very fast solvers have opened the possibility of handling very large volume elements or inverse problems, e.g., for finding optimal particle shapes for given load cases and damage modes (Zohdi, 2003). Further increases in efficiency can be achieved for very large voxel-based linear DIB models by exploiting the fact that all elements pertaining to a given phase have identical element stiffness matrices (in linear regimes).

Special Finite Element formulations (and dedicated Finite Element programs) are typically required for asymptotic homogenization models, compare section 5.3, and for schemes that concurrently solve the macroscopic and microscopic problems (Urbański, 1999).

A further, relatively recent numerical approach that is related to FE methods are reduced basis (or reduced order) homogenization schemes, see, e.g., Fritzen and Kunc (2018), which use globally supported trial and weighting functions to obtain much smaller algebraic systems. Appropriately choosing such functions and performing efficient numerical integration on them are important issues for such reduced order methods. The Nonuniform Transformation Analysis mentioned earlier was among the first methods of this type.

Discrete microstructure approaches employing numerical engineering methods are best suited to studying phase arrangements in which the characteristic lengths of the important geometrical features do not differ excessively. If this is not the case, e.g., for randomly oriented inhomogeneities of high aspect ratio, very large volume elements may be required,

\(^6\)Essentially, in such a scheme the initial guess gives a good estimate of “long wavelength” contributions to the solution, and the CG iterations take care of “short wavelength” variations.
meshing may become onerous, and the numerical effort for solving the models may become very high, indeed, especially for three-dimensional configurations. For methods making use of structured meshes, such as FFT algorithms and FE-based voxel models, the chosen mesh or voxel size intrinsically limits geometrical resolution. In the case of approaches that employ unstructured meshes and support local mesh refinement, such as standard FE methods in the sense of figure 4.2, practical limits on resolution tend to be imposed by the size and conditioning of the resulting systems of algebraic equations. For example, discrete microstructure models of composites reinforced by polydispersely sized particles the diameters of which differ by more than a factor of, say, 10, tend to become very unwieldy. Another issue requiring special precautions is the handling of extremely closely spaced reinforcements in full field models\(^{61}\), see, e.g., Gusev (2016).

Under conditions of macroscopic softening, e.g., due to damage or localization, discretizing methods are liable to producing mesh-dependent results. Such tendencies may be counteracted by appropriate regularization procedures, among them enrichment with higher-order gradients, see e.g., Geers et al. (2001a), nonlocal averaging of the rate of an appropriate internal variable, compare, e.g., Jirásek and Rolshoven (2003), the use of time dependent formulations involving rate effects, see, e.g., Needleman (1987), or local averaging, compare, e.g., Fang et al. (2016).

### 4.4 Evaluation of Results

When linear elastic or thermoelastic inhomogeneous materials are studied, the major aim of homogenization in full field approaches consists in evaluating the effective elasticity tensors, \( \mathbf{E}^* \), and thermal expansion tensors, \( \alpha^* \). In homogenization studies involving non-linear, path-dependent constituent behavior, e.g., elastoplasticity, thermoelastoplasticity, viscoelastoplasticity or damage, in contrast, no comparably general solutions exists. Here, typically the evolution of appropriate macroscopic variables is followed along some specified load path, e.g., in the form of effective stress–strain relations\(^{62}\).

In linear elastic homogenization it is typically preferable to solve for the macroscopic elastic tensors, which automatically provide full information on macroscopic elastic symmetry of the composite, rather than reconstructing them from results on individual moduli\(^{63}\). For three-dimensional configurations, six linearly independent mechanical load cases must be applied to the volume element for evaluating macroscopic elastic tensors, and a homogeneous temperature change is required for obtaining the macroscopic thermal expansion tensor. The effective elastic tensors of inhomogeneous materials with orthotropic or higher

---

\(^{61}\)Strict two-phase configurations may, in fact, constitute a considerable idealization for the regions close to touching (“percolating”) or nearly touching inhomogeneities, because production processes may give rise to preferential accumulation of pores and impurities at such positions in actual materials.

\(^{62}\)Homogenized tangent tensors may also be evaluated for given load states, compare, e.g., (Ghosh et al., 1996). This is of special interest in hierarchical and multi-scale analysis employing discrete microstructure models as implicit material models for Finite Element models on the macroscale.

\(^{63}\)Ensemble averaging over a number of SVEs also is arguably best done in terms of the elastic tensors. Furthermore, if load-controlled periodic homogenization is used, elastic tensors can be obtained at moderate cost because solutions for the required linearly independent load vectors can be found cheaply once the system matrix has been factorized.
symmetry show similar structures to those of isotropic symmetry, see eqns. (1.12) and (1.13), i.e., in Voigt/Nye notation all tensor elements are zero with the exception of the upper left submatrix and the diagonal of the lower right submatrix. In contrast, volume elements obtained from real structures or generated with stochastics-based models are typically too small to be proper RVEs and, accordingly, tend to introduce small contributions from lower elastic symmetries. In addition, minor perturbations due to roundoff errors may be present.

Accordingly, volume elements aimed at describing statistically isotropic, transversely isotropic or orthotropic material behavior must be expected to actually return lower macroscopic material symmetries. Such deviations from an “elastic target symmetry”, which tend to decrease with growing size (and number, presuming the use of ensemble averaging) of SVEs, can be corrected to some extent by finding the elastic tensor of the required symmetry which is closest to the “raw” elastic tensor evaluated from homogenization. Pahr and Böhm (2008) proposed a simplified approach in which the “raw” elastic tensor resulting from the homogenization of an SVE of the type shown in fig. 5.4, which aims at describing a statistically isotropic composite, is first “orthotropized” by setting all sub-orthotropic terms to zero and, if necessary, symmetrized. Closest isotropic tensors can then be obtained as the isotropic term of a generalized spherical harmonics expansion of distributions functions of the orthotropic elongation and bulk moduli, using a procedure developed by He and Curnier (1997). Alternatively, expressions resulting from the minimization of an appropriately defined distance between an elasticity tensor of prescribed symmetry and a given anisotropic elasticity tensor, as discussed by Norris (2006), Moakher and Norris (2006) as well as Bucataru and Slawinski (2009), may be made use of for finding the closest isotropic or transversally isotropic elasticity tensors. Such procedures have been successfully applied to evaluating effective moduli pertaining to macroscopically isotropic elastic behavior from ensemble averaged sets of SVEs by Rasool and Böhm (2012), giving excellent agreement with the three-point estimates of of Torquato (1997, 1998a) for randomly positioned, non-interpenetrating, identical spherical reinforcements, compare section 2.4. An alternative way of “isotropizing” the anisotropic tensors resulting from homogenizing SVEs involves isotropic averaging, compare, e.g., Cook and Young (1985); for a study employing a “rotation averaging” approach see, e.g., Gusev (2016).

In micromechanical applications numerical engineering methods directly evaluate the microscopic fields, i.e., for mechanical analyses, the fluctuating displacements, stresses and strains as well as fields derived from them. In the case of displacement-based FE methods using isoparametric elements, the microscopic stresses and strains are primarily evaluated at the integration point level of the individual elements, nodal values being obtained by extrapolation and averaging procedures. Some care, however, is necessary in using such data, because the extreme values of microscopic stresses and strains may depend markedly on details of the microgeometry, of the discretization used and, in the case of nodal data,
of the evaluation procedure. Furthermore, idiosyncrasies of the numerical methods may come into play, one case being the reduced accuracy of stress and strain fields obtained by displacement-based FE methods at interfaces between constituents and at the models’ surfaces compared to the interior of regions occupied by a given phase.

For evaluating volume averages of stress- and strain-like variables from numerical predictions for microscopic fields in a small strain setting, it is typically good practice to use direct volume integration on the basis of eqn. (1.16). Many FE codes provide the data necessary for approximating volume averaging by approximate numerical quadrature according to

$$
\langle f \rangle = \frac{1}{\Omega} \int_{\Omega} f(z) d\Omega \approx \frac{1}{\Omega} \sum_{l=1}^{N} f_l \Omega_l.
$$  \hspace{1cm} (4.8)

Here $f_l$ and $\Omega_l$ are the function value and the integration weight (in terms of the volume of the integration point), respectively, associated with the $l$-th integration point within a given integration volume $\Omega$ that contains $N$ integration points. By analogy, the standard deviation of function $f(z)$ over volume $\Omega$ can be evaluated as

$$
S(f) \approx \sqrt{\frac{1}{\Omega} \sum_{l=1}^{N} (f_l - \langle f \rangle)^2 \Omega_l}.
$$  \hspace{1cm} (4.9)

When phase averages are to be generated of variables that are nonlinear functions of the stress and strain components (e.g., equivalent stresses, equivalent strains, stress triaxialities), only direct volume averaging of these variables may be used, because evaluating nonlinear variables from the averaged components may lead to unacceptable inaccuracies, compare section 2.5. When suitably fine meshes are employed, eqn. (4.8) typically returns accurate results in the small strain regime and useful approximations in other cases.

Besides generating overall and phase averages, microscopic variables can also be evaluated in terms of averages and standard deviations in individual inhomogeneities, which supports assessing inter- and intra-inhomogeneity fluctuations of the stress and strain fields, compare the fiber level averages and standard deviations of the maximum principal stress displayed in fig. 2.4.

In addition, distribution functions (“stress spectra”, see, e.g., Bornert et al. (1994), Böhm and Rammerstorfer (1995) or Böhm and Han (2001)) can be extracted from discrete...
microstructure models for the whole composite, for a given phase or for individual inhomogeneities. The predictions for the tail regions of such distributions, however, tend to show at least some dependence on the mesh and discretization used, so that care is typically required in interpreting them.
Chapter 5

Periodic Microfield Models

Periodic Microfield Approaches (PMAs) aim at approximating the macroscopic and microscopnic behavior of inhomogeneous materials by studying model materials that have periodic microstructures.

5.1 Basic Concepts of Periodic Models

Periodic microfield approaches analyze the behavior of infinite (one, two- or three-dimensional) periodic phase arrangements under the action of far field mechanical loads or uniform temperature fields, for which the Hill–Mandel criterion can be fulfilled as discussed in section 4.2. The most common approach to studying the stress and strain fields in such periodic configurations is based on describing the microgeometry by a periodically repeating unit cell (RUC), to which the investigations may be limited without loss of information or generality, at least for static analysis.

Standard PMAs cannot handle macroscopic gradients in mechanical loads, temperature or composition in any direction in which periodicity of the fields is prescribed. Such gradients or boundaries can be studied, however, in directions where periodicity is not prescribed, a typical case being layer-type models that are non-periodic in one direction and periodic in the other(s), see, e.g., Wittig and Allen (1994), Reiter et al. (1997) and Weissenbek et al. (1997).

In the present report the designation “unit cell” is used for any volume element that can generate a periodic microgeometry. Accordingly, a unit cell may comprise a simple periodic base unit (or part of it), a collective of simple periodic base units, or a phase arrangement of arbitrary geometrical complexity (multi-fiber or multi-particle unit cell) that shows translational periodicity; in the limiting case a unit cell may thus be a proper representative volume element. Accordingly, the discussion of volume element sizes in section 4.1 is fully pertinent to unit cell models.

Periodic phase arrangements are not very well suited to modeling the transient behaviors of structural composite materials. In dynamic settings, RUC act as phononic crystals, i.e., as metamaterials exhibiting phononic band structures and dispersion relations. Studying such behavior, however, is a research field in its own right, compare, e.g., Hussein et al. (2014), in which periodic volume elements play an important role, see, e.g., Suzuki and Yu (1998).

Unit cells are typically limited to describing wavelengths smaller than or equal to their relevant dimension, which is a direct consequence of the boundary conditions required for obtaining periodicity. By analogy, in stability analysis of inhomogeneous materials RUCs can directly resolve only buckling modes of specific wavelengths, so that considerable care is required in using them, compare Vonach (2001) or Pahr and Rammerstorfer (2006). However, bifurcation modes with wavelengths exceeding the size of the unit cell can be handled via periodic linear models by using the Bloch theorem (Gong et al., 2005).
The literature on periodic homogenization of inhomogeneous materials is fairly extensive, and well developed mathematical theories are available on scale transitions in periodic structures and materials, compare Michel et al. (2001). A wide variety of unit cells have been employed in such studies, ranging from geometries that describe simple periodic arrays (“lattices”) of inhomogeneities to complex periodic phase arrangements, such as volume elements containing considerable numbers of statistically arranged inhomogeneities. For some simple periodic phase arrangements and for linear material behavior it has proven possible to find analytical solutions based on series expansions making direct use of the periodicity (Sangani and Lu, 1987; Wang et al., 2000; Cohen, 2004; Drago and Pindera, 2008).

Even though most PMA studies in the literature have used standard numerical engineering methods as discussed in section 4.3, some other numerical schemes have been proposed that are specialized to periodic phase arrangements. One of them, known as the Method of Cells (Aboudi, 1989, 1991), in its basic form discretizes unit cells that correspond to square arrangements of square fibers into four subcells, within each of which displacements are approximated by low-order polynomials. Traction and displacement continuity conditions at the faces of the subcells are imposed in an average sense and analytical and/or semi-analytical approximations to the deformation fields are obtained in the elastic and inelastic ranges. Because they use highly idealized microgeometries, provide only limited information on the microscopic stress and strain fields, and have limited capabilities for handling axial shear, the resulting models pose relatively low computational requirements and can provide constitutive descriptions for analyzing structures made of continuously reinforced composites, see, e.g., Arenburg and Reddy (1991). Developments of the algorithm led to the Generalized Method of Cells (Aboudi, 1996), which is more flexible geometrically and allows finer discretizations of unit cells for fiber and particle reinforced composites, reinforcement and matrix being essentially split into a number of “subregions” of rectangular or hexahedral shape. For some comparisons with microfields obtained by Finite Element based unit cells see, e.g., Iyer et al. (2000) or Pahr and Arnold (2002). Later, higher order displacement interpolants were brought in to obtain the High-Fidelity Generalized Method of Cells (Aboudi, 2004), which provides an alternative to Finite Element algorithms for “pixel element” micromechanical models (compare section 4.3). This method was also extended to general quadrilateral cell shapes (Haj-Ali and Aboudi, 2010).

A further group of solution strategies for PMAs reported in the literature (Axelsen and Pyrz, 1995; Fond et al., 2001; Schjødt-Thomsen and Pyrz, 2004) use numerically evaluated equivalent inclusion approaches that account for interacting inhomogeneities as provided, e.g., by the work of Moschovidis and Mura (1975). Multipole expansion methods can handle complex periodic matrix–inclusion microgeometries (Sangani and Mo, 1997; Kushch et al., 2008). Alternatively, the elastic fields in periodic inhomogeneous materials can be evaluated numerically via variational methods for determining stress-free strain fields (Wang et al., 2002).

As mentioned in section 1.5, periodic homogenization cannot be freely used for modeling damage in and failure of inhomogeneous materials, all relevant geometrical features, among them crack patterns, being per definition periodic. Consequently, PMAs cannot describe behavior involving macroscopic localization of damage, failure in periodic models rather being akin to some kind of fragmentation. This, however, does not rule out the use
of PMAs for modeling smeared-out damage in parts of larger models, e.g., within domain-splitting schemes as used by Raghavan and Ghosh (2004).

For developing periodic microfield approaches the strain and stress fields are typically decomposed into constant macroscopic strain and stress contributions (“slow variables”), $\langle \varepsilon \rangle$ and $\langle \sigma \rangle$, plus periodically varying microscopic fluctuations (“fast variables”), $\varepsilon'(z)$ and $\sigma'(z)$, compare section 4.2. Furthermore, it is convenient to introduce a “microscopic coordinate” $z$ that is scaled such that it can resolve the local features of the unit cell.

An idealized depiction of periodic microfields is presented in fig. 5.1, which shows the variations of the strains $\varepsilon_s(z) = \langle \varepsilon_s \rangle + \varepsilon'_s(z)$ and of the corresponding displacements $u_s(z) = \langle \varepsilon_s \rangle z + u'_s(z)$ along some line $z$ in a hypothetical one-dimensional periodic two-phase material consisting of constituents A and B, the length of the unit cell (“unit of periodicity”) being $c_z$. The volume averaged strain, $\langle \varepsilon_s \rangle$, is linked to the displacement increment per unit cell, $\Delta u_s$, by the relations

$$\langle \varepsilon_s \rangle = \frac{\Delta u_s}{c_z} \quad \text{and} \quad u_s(z + c_z) = u_s(z) + \langle \varepsilon_s \rangle c_z . \quad (5.1)$$

The periodicity of the strains and the cumulative nature of the displacements are evident.

![Figure 5.1: Schematic depiction of the variation of the strains $\varepsilon_s(z)$ and the displacements $u_s(z)$ along a generic “one-dimensional periodic composite” of constituents A and B with unit of periodicity $c_z$. Symmetry points of $\varepsilon_s(z)$ and $u_s(z)$ are indicated by small circles.](image)

### 5.2 Boundary Conditions

Unit cells together with the boundary conditions (B.C.s) prescribed on them must generate valid tilings of the undeformed geometry and as well as for all deformed states pertinent
to a given micromechanical problem. Accordingly, gaps and overlaps between neighboring volume elements as well as unphysical constraints on their deformations must not be allowed, i.e., the cells must be geometrically compatible. In order to achieve this, the boundary conditions for the unit cells must be specified in such a way that all deformation modes appropriate for the load cases to be studied can be attained. The major types of boundary conditions used in periodic microfield analysis are periodicity, symmetry, and antisymmetry B.C.s\textsuperscript{70}. In PMA models one of these three types of boundary conditions (or a combination of them) must be used on any exterior boundary of the volume element, irrespective of the numerical method employed for solving the equilibrium equations.

Generally, for any given periodic phase arrangement unit cells are non-unique, the range of possible shapes being especially wide when point or mirror symmetries are present in the microgeometry (as tends to be the case for regular lattices). As an example, fig. 5.2 depicts a (two-dimensional) periodic hexagonal array of circular inhomogeneities (e.g., fibers oriented normally to the plane) and some of the unit cells that can be used to study aspects of the behavior of this phase arrangement. There are considerable differences in the sizes and capabilities of the unit cells shown.

![Figure 5.2: Periodic hexagonal array of circular inhomogeneities in a matrix and 10 unit cells or “reduced cells” that can be used to describe the mechanical responses of this arrangement under loads acting parallel to the coordinate axes.](image)

Periodicity Boundary Conditions

The most general boundary conditions for volume elements in periodic homogenization are periodicity (periodic fluctuation, “toroidal”, “cyclic”) B.C.s, which can handle any physically valid deformation state of the unit cell and, consequently, of the inhomogeneous

\textsuperscript{70}For more formal treatments of boundary conditions for unit cells than given here see, e.g., Anthoine (1995) or Michel et al. (1999).
material to be modeled. Periodicity boundary conditions make use of translatory symmetries of a given geometry; in fig. 5.2 cells A to E belong to this group.

In order to describe an \( N \)-dimensional phase arrangement with translatory periodicity, a suitable unit cell and a set of \( N \) linearly independent periodicity vectors \( \mathbf{p}_n \) are required. These periodicity vectors are neither unique nor do they have to be orthogonal. For any given periodic microgeometry the minimum volume of pertinent unit cells is well defined, but such “minimum unit cells” can take an infinite number of shapes, some examples being shown for a simple two-dimensional case in fig. 5.3. The surface of any unit cell to be used with periodicity boundary conditions must consist of at least \( N \) pairs of faces (or pairs of parts of faces) \( \Gamma_k \), and the surface elements making up a given pair, \( k^- \) and \( k^+ \), must be identical but shifted relative to each other by “shift vectors” \( \mathbf{c}_k \). Each shift vector, in turn, must be a linear combination of the periodicity vectors, i.e., \( \mathbf{c}_k = \sum m^k_l \mathbf{p}_l \), where the \( m^k_l \) are integer numbers. In fig. 5.3 matching pairs of faces (or, in the case of some cells, parts of faces) \( \Gamma_k \) are marked by being drawn in identical color and line style. Obviously, faces of unit cells may be curved, compare, e.g., Garnich and Karami (2004).

![Figure 5.3: Eight different but equivalent periodic minimum-size unit cells for a two-dimensional periodic matrix–inclusion medium with two (slightly) non-orthogonal periodicity vectors \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) (\( \mathbf{p}_1' \) and \( \mathbf{p}_2' \) form an alternative pair of periodicity vectors). Paired faces (or parts of faces) \( \Gamma_k \) are marked by identical line styles as well as colors.](image)

The selection of the shape of unit cells involves trade-offs: On the one hand, unit cells of simple shape facilitate the application of periodicity boundary conditions at least to some extent. On the other hand, low-angle intersections between phase boundaries and cell faces as well as phase boundaries that closely approach cell faces often make such cells difficult to mesh for FE analysis. The latter issue can be alleviated or avoided by choosing suitably shaped and positioned unit cells. The process of defining such cells can be automated, e.g., by using modified Voronoi-type algorithms, which, however, leads to volume elements of irregular shape such as the leftmost unit cell in fig. 5.3. In practice, computer generated periodic volume elements have typically been set up such that the periodicity vectors are orthogonal, which facilitates generating unit cells that are rectangles or right hexahedra and supports straightforward naming schemes of faces, edges, and vertices.
In the following a nomenclature is used in which the faces of two-dimensional quadrilateral unit cells are denoted as N, S, E and W (for North, South, East, and West which are used as in topographical maps), vertices being named according to the adjoining cell faces, compare figs. 5.5 to 5.7. The faces of three-dimensional cells of hexahedral shape are, by analogy, referred to as N, S, E, W, B and T (the latter standing for bottom and top), and edges as well as vertices are referred to via the adjoining faces (e.g., SE or SWB), see fig. 5.4.

![Cube-shaped periodic unit cell](image)

Figure 5.4: Cube-shaped periodic unit cell containing 15 randomly positioned spherical particles of equal size at a volume fraction of $\xi^{(i)} = 0.15$. Designators of the six faces (East, West, North, South, Top, Bottom) and of the vertices are given (Pahr and Böhm, 2008).

On the basis of the above discussion of unit cell geometries, eqn. (5.1) can be expanded into the expression

$$u(z + c_k) = u(z) + \langle \varepsilon \rangle * c_k$$

(5.2)

for multi-dimensional cases, $c_k$ being an appropriate shift vector. The unit cells tile the computational space by translation, so that neighboring cells (and, consequently, the “opposite” faces of a given cell) must fit into each other like the pieces of a jigsaw puzzle in both undeformed and deformed states. For each pair of surface elements, $\Gamma_k$, eqn. (5.2) allows expressing periodicity boundary conditions for the mechanical problem in the small strain regime as

$$\Delta u_k = u_{k^+} - u_{k^-} = u(s_k + c_k) - u(s_k) = \langle \varepsilon \rangle * c_k,$$

(5.3)

where $u_{k^-}$ and $u_{k^+}$ are the displacements at pairs of homologous nodes positioned at $s_k$ and $s_k + c_k$, respectively, on the surface elements $k^-$ and $k^+$ (which may, e.g., correspond to faces N and S in figs. 5.4 and 5.5). The vector linking such pairs of nodes in a deformed state is $\hat{c}_k = c_k + \Delta u_k$. The macroscopic strain $\langle \varepsilon \rangle$ according to eqn. (5.3) is prescribed in displacement controlled analysis and is a macroscopic response in load controlled analysis. Equation (5.3) allows expressing the displacement vector $u_{k^+}$ as

$$u_{k^+} = u_{k^-} + u_{M^+} - u_{M^-},$$

(5.4)
where $u_{M+}$ and $u_{M-}$ are the displacements of an appropriate pair of homologous “master nodes”, which may (but do not have to) be chosen among the vertices of the periodic volume element. These control nodes carry the information on $\hat{c}_k$. Such constraint conditions enforce a seamless fit between neighboring unit cells for all possible deformed states.

For the special case of initially rectangular two-dimensional unit cells, such as the one shown in fig. 5.5, eqns. (5.4) lead to the expressions

$$u_N(\tilde{s}_1) = u_S(\tilde{s}_1) + u_{NW} \quad \text{and} \quad u_E(\tilde{s}_2) = u_W(\tilde{s}_2) + u_{SE}, \quad (5.5)$$

where the vertex SW ("M-" in the above scheme) is chosen to be fixed and $\tilde{s}_k$ are local “face coordinates” that are used to denote homologous nodes on pairs of faces. Note that eqns. (5.5) directly imply that

$$u_{NE} = u_{NW} + u_{SE}.$$  

For numerical analysis the two faces making up a homologous pair $\Gamma_k$ are best discretized in a compatible way, so that the nodal points on them are positioned at identical values of the “face coordinates” $\tilde{s}_k$, ensuring that they are separated by a shift vector $c_l$. Equations (5.3) then become sets of linear constraints each of which links three nodal displacement DOFs$^{71}$.

Comparing eqns. (5.3) and (5.5) shows that the displacements of the “master nodes”, SE and NW, contain the information on the macroscopic strain tensor $\langle \varepsilon \rangle$. In addition, the displacements of the master nodes and of faces S and W fully control the displacements of the “slave faces” N and E.

Figure 5.5: Sketch of periodicity boundary conditions as used with an initially rectangular two-dimensional unit cell.

Conditions analogous to eqn. (5.5) can be specified for any periodic, space-filling and regular two-dimensional cell that has an even number of sides (e.g., squares, rectangles, or

$^{71}$In principle, all variables (i.e., for mechanical analysis the displacements, strains and stresses) must be linked by appropriate periodicity conditions (note that, in contrast to stresses, boundary traction vectors are antiperiodic). When a displacement based FE code is used such conditions can be specified explicitly only for the displacement components (including, where appropriate, rotational DOFs), the implied natural traction B.C. giving rise to antiperiodic tractions in such a setting (Li, 2012). Usually, however, the periodicity of nodal tractions is fulfilled only approximately. It is also worth noting that the averaging procedures typically employed for evaluating nodal do not account for periodicity linkage in typical FE implementations.
hexagons) or three-dimensional cell that has an even number of faces (e.g., cubes, hexahedra, rhombic dodecahedra, or regular tetrakaidecahedra) arranged such that they form homologous pairs. The scheme can also be extended to unit cells of less regular shape provided their faces can be suitably paired off, compare Cruz and Patera (1995), Estrin et al. (1999) or Xia et al. (2003) as well as figs. 5.2 and 5.3. Periodicity B.C.s generally are the least restrictive option for multi-inhomogeneity unit cell models using phase arrangements generated by statistics-based algorithms. Compared to other discrete microstructure approaches, periodic homogenization typically shows the fastest convergence in terms of sizes of volume elements, see, e.g., El Houdaiguì et al. (2007).

For layer-type models, which can be generated by specifying free surfaces at appropriate pairs of boundaries, macroscopic rotational degrees of freedom, i.e., macroscopic bending and twisting, can be studied by appropriately modifying the standard periodicity B.C.s, eqns. (5.2) to (5.5), for the other faces. When such “structure-type” macroscopic rotational degrees of freedom are present, out-of-plane shear loads must be accompanied by appropriate bending moments to achieve stress equilibrium, compare, e.g., Urbaníski (1999). Models of this type models allow using periodic homogenization for evaluating homogenized “laminate stiffnesses” of inhomogeneous plates and shells for use within classical lamination theory (Jones, 1999). For detailed information see, e.g., Pahr and Rammerstorfer (2006).

The above treatment can be carried over into the large strain regime. Periodicity boundary conditions that are conceptual developments of eqn. (5.5) can be devised for cases where standard conditions for homogenization are not met, gradient (nonlocal) theories are employed on the macroscale, and higher-order stresses as well as strain gradients figure in coupling the length scales (Geers et al., 2001b). The resulting deformation patterns of the cells, however, no longer follow geometrical compatibility as discussed above.

In practice FE-based studies of volume elements subjected to periodicity boundary conditions can be somewhat expensive, especially in terms of providing the required compatible discretizations on periodic pairs of faces, which can be difficult to automatize. Accordingly, there has been research interest in relaxing the compatibility requirements, see, e.g., Wippler et al. (2011), in implementing periodicity B.C. for non-compatible meshes on paired faces via constraint equations, compare, e.g., Nguyen et al. (2012), or in applying periodicity B.C. to non-periodic volume elements, see, e.g., Schneider et al. (2017). Such approximations typically give rise to some perturbations of the microfields close to the cell boundaries that are related to the boundary layers shown by windowing and embedding models, compare section 4.2. The strength of these perturbations depends on the algorithm used, and they tend to have little to limited influence on the predicted homogenized behavior in linear and weakly nonlinear regimes. For strongly nonlinear behaviors, such as phase-level damage, however, spurious local fields due to boundary perturbations may markedly influence the predicted macroscopic responses, even for phase arrangements closely approaching representativity for linear properties.
Symmetry Boundary Conditions

For rectangular and hexahedral volume elements in which the faces of the unit cell coincide with symmetry planes of the phase arrangement and for which this property is retained for all deformed states that are to be studied, periodicity B.C.s simplify to symmetry (or mirror) boundary conditions over a subset of the periodic volume, called a “reduced cell” in the following. Following the nomenclature of fig. 5.6 these B.C.s take the form

\[
\begin{align*}
  u_E(\tilde{s}_2) &= u_{SE} & v_N(\tilde{s}_1) &= v_{NW} & u_W(\tilde{s}_2) &= 0 & v_S(\tilde{s}_1) &= 0,
\end{align*}
\]

where \(u\) and \(v\) stand for the displacement components in 1- and 2-direction, respectively. Equation (5.6) puts constraints on the normal displacement components at the reduced cells’ surfaces, but leaves the tangential displacements free, thus enforcing the condition that pairs of opposite faces must stay parallel throughout the deformation history. Accordingly, symmetry boundary conditions do not allow fluctuations of the local fields in directions normal to a given face of a unit cell or reduced cell\(^72\). For symmetry B.C.s there are no requirements with respect to compatibility of phase arrangements or meshes at different faces.

![Figure 5.6: Sketch of symmetry boundary conditions as used with a rectangular two-dimensional reduced cell.](image)

Symmetry boundary conditions are fairly easy to use and tend to give rise to small reduced cells for simply periodic phase arrangements. However, the load cases that can be handled are limited to uniform thermal loads, mechanical loads that act in directions normal to one or more pairs of faces, and combinations of the above\(^73\). In fig. 5.2 cell F uses boundary conditions of this type. Symmetry boundary conditions are typically very useful for describing relatively simple microgeometries, but impose marked limitations on more complex phase arrangements.

---

\(^{72}\)Symmetry with respect to a given face implies that the normal (but not the tangential) components of the fluctuations of the microfields, compare eqns. (1.2) and (4.6), must vanish there. The stress and strain fluctuations \(\varepsilon'(z)\) and \(\sigma'(z)\) must obviously be symmetric with respect to symmetry planes, whereas displacements accumulate across symmetry planes, compare fig. 5.1.

\(^{73}\)Symmetry boundary conditions are compatible with uniaxial stress, uniaxial strain, extensional shear (obtained in the two-dimensional case by applying normal stresses \(\sigma_a\) to the vertical and \(-\sigma_a\) to the horizontal faces), hydrostatic and (hygro)thermal load cases. Accordingly, symmetry B.C.s are often sufficient for materials characterization. However, the normal and shear stresses evaluated as above do not pertain to the same material coordinate system, extensional shear corresponds to pure shear in a coordinate system rotated by 45°.
Antisymmetry Boundary Conditions

Antisymmetry (point symmetry, central reflection) boundary conditions require the presence of centers of point symmetry ("pivot points") and are, accordingly, even more limited in terms of the microgeometries that they can handle. In contrast to symmetry boundary conditions, however, unit cells employing antisymmetry B.C. on all faces are subject to few restrictions in terms of the load cases that can be handled. Among the volume elements shown in fig. 5.2, cells G and H use point symmetry B.C.s on all faces and can handle any in-plane deformation\(^{74}\). Alternatively, antisymmetry B.C.s can be combined with symmetry B.C.s to obtain very small reduced cells that are restricted to loads acting normal to the symmetry faces, compare cells I and J in fig. 5.2. Figure 5.7 shows such a reduced cell, the antisymmetry boundary conditions being applied on face E where a pivot point P is present. For this configuration the boundary conditions

\[
\begin{align*}
  u_U(\tilde{s}_P) + u_L(-\tilde{s}_P) &= 2u_P, \\
  v_N(\tilde{s}_1) &= v_{NW} = 2v_P, \\
  v_S(\tilde{s}_1) &= 0, \\
  u_W(\tilde{s}_2) &= 0,
\end{align*}
\]

must be fulfilled, where \(u_U(\tilde{s}_P)\) and \(u_L(-\tilde{s}_P)\) are the displacement vectors of pairs of homologous nodes U and L that are positioned symmetrically with respect to the pivot point P. As indicated in fig. 5.7 the local coordinate system \(\tilde{s}\) is defined on face E and centered on P. The undeformed geometry of such a face also must be antisymmetric with respect to the pivot point P, and the phase arrangements as well as the discretizations on both halves of face E must be compatible. Three-dimensional reduced cells employing combinations of symmetry and point symmetry B.C.s can, e.g., be used to advantage for studying cubic arrays of particles, see Weissenbek et al. (1994), or woven composites, compare Li and Zou (2011).

\(^{74}\)Unit cells and reduced cells using antisymmetry boundary conditions may have odd numbers of faces. Triangular volume elements similar to cell H in fig. 5.2 were used, e.g., by Teply and Dvorak (1988) to study the transverse mechanical behavior of hexagonal arrays of fibers. Rectangular cells with point symmetries on each boundary were introduced by Marketz and Fischer (1994) for perturbed square arrangements of inhomogeneities. The periodic cells D and E also show point symmetry on their faces.
For a reduced cell of the type shown in fig. 5.7, either the displacements of the master nodes SE and NW, $u_{SE}$ and $u_{NW}$, or the displacements of the pivot P, $u_P$, can be used for evaluating the macroscopic strain of the corresponding periodic model material, compare eqn. (5.17).

The description of real materials, which in general are not periodic, by periodic model materials may entail some geometrical approximations. In the case of real structure geometries this may take the form of applying periodicity constraints to non-periodic geometries or modifying the geometries such that they become periodic. For periodic volume elements approaching being proper RVEs, the resulting errors in terms of the homogenized responses will be of little practical relevance. However, perturbations of the microfields in the vicinity of the cell surfaces will be present, which should be monitored in the case of highly nonlinear analysis. In the linear range the use of windowing methods with appropriate mixed boundary conditions, compare chapter 6, provides a superior alternative to ad-hoc “periodification” procedures.

Finally, it is worth noting that all of the above boundary conditions, eqns. (5.4), (5.5), (5.6) and (5.7) as well as their three-dimensional equivalents, can be handled by any FE code that provides linear multipoint constraints between degrees of freedom and thus allows the linking of three or more D.O.F.s by linear equations.

5.3 Application of Loads and Evaluation of Fields

Once suitable volume elements have been defined and appropriate boundary conditions applied, the appropriate loads in the form of uniform macroscopic stresses as well as strains and/or homogeneous temperature excursions must be applied, i.e., the microscopic and macroscopic fields must be linked. Whereas loading by uniform temperature increments does not pose major difficulties, applying far field stresses or strains is not necessarily straightforward. For example, the variations of the stresses along the faces of a unit cell in general are not known a priori, so that it is not possible to prescribe the boundary tractions via distributed loads. There are two major approaches to implementing the micro–macro linkage.

Asymptotic Homogenization

The most versatile and elegant strategy for linking the macroscopic and microscopic fields in periodic microfield models is based on a mathematical framework referred to as asymptotic homogenization, asymptotic expansion homogenization, or homogenization theory, see, e.g., Suquet (1987) or Auriault (2002). It is based on explicitly introducing macroscopic and microscopic coordinates, $y$ and $z$, respectively, into the formulation of the problem.

---

75In the same vein, non-periodic, rectangular or cube-shaped volume elements may simply be subjected to symmetry B.C. or may be mirrored with respect to two or three of their faces and then subjected to periodicity B.C. These operations, however, tend to lead to undesirable shapes of reinforcements and to changes in the phase arrangement statistics. Nevertheless, for sufficiently large multi-inhomogeneity cells useful results can be obtained with such a strategy, see, e.g., (Terada et al., 2000).
Microscopic and physical coordinates are linked by the expression

\[ z_i = x_i / \epsilon \]  

(5.8)

where \( \epsilon = \ell / L \ll 1 \) is a scaling parameter. \( L \) and \( \ell \) stand for the characteristic lengths of the macro- and microscales\(^{76}\). The displacement field in the unit cell can then be represented by an asymptotic expansion of the type

\[ u_i(y, z, \epsilon) = u_i^{(0)}(y) + \epsilon u_i^{(1)}(y, z) + \epsilon^2 u_i^{(2)}(y, z) + \text{H.O.T.} \]  

(5.9)

where the \( u_i^{(0)} \) are the effective or macroscopic displacements and \( u_i^{(1)} \) stands for the periodically varying displacement fluctuations due to the microstructure\(^{77}\).

Using the chain rule, i.e.,

\[ \frac{\partial}{\partial x} f(y(x), z(x), \epsilon) \rightarrow \frac{\partial}{\partial y} f + \frac{1}{\epsilon} \frac{\partial}{\partial z} f \]  

(5.10)

in the small strain regime the strains can be related to the displacements as

\[ \varepsilon_{ij}(y, z, \epsilon) = \frac{1}{2} \left\{ \left( \frac{\partial}{\partial y_j} u_i^{(0)} + \frac{\partial}{\partial y_i} u_j^{(0)} \right) + \left( \frac{\partial}{\partial z_j} u_i^{(1)} + \frac{\partial}{\partial z_i} u_j^{(1)} \right) \right\} + \epsilon \left\{ \left( \frac{\partial}{\partial y_j} u_i^{(1)} + \frac{\partial}{\partial y_i} u_j^{(1)} \right) + \left( \frac{\partial}{\partial z_j} u_i^{(2)} + \frac{\partial}{\partial z_i} u_j^{(2)} \right) \right\} + \text{H.O.T.} \]  

\[ = \varepsilon_{ij}^{(1)}(y, z) + \epsilon \varepsilon_{ij}^{(2)}(y, z) + \text{H.O.T.} \]  

(5.11)

where terms of the type \( \varepsilon_{ij}^{(0)} = \frac{1}{\epsilon} \frac{\partial}{\partial z_i} u_i^{(0)} \) are deleted due to the underlying assumption that the variations of slow variables are negligible at the microscale. By analogy the stresses can be expanded into the expression

\[ \sigma_{ij}(y, z, \epsilon) = \sigma_{ij}^{(1)}(y, z) + \epsilon \sigma_{ij}^{(2)}(y, z) + \text{H.O.T.} \]  

(5.12)

Using the two-scale assumption and, as a consequence, eqn. (5.10), the equilibrium equations take the form

\[ \left( \frac{\partial}{\partial y_j} + \frac{1}{\epsilon} \frac{\partial}{\partial z_j} \right) \sigma_{ij}(y, z, \epsilon) + f_i(y) = 0 \]  

(5.13)

the \( f_i \) being macroscopic body forces. By inserting eqn. (5.12) into this expression and sorting the resulting terms by order of \( \epsilon \) a hierarchical system of partial differential equations is obtained, the first two of which are

\[ \frac{\partial}{\partial z_j} \sigma_{ij}^{(1)} = 0 \quad \text{(order } \epsilon^{-1}) \]  

\[ \frac{\partial}{\partial y_j} \sigma_{ij}^{(1)} + \frac{\partial}{\partial z_j} \sigma_{ij}^{(2)} + f_i = 0 \quad \text{(order } \epsilon^0) \]  

(5.14)

\(^{76}\)Equation (5.8) may be viewed as “stretching” the microscale so it becomes comparable to the macroscale, \( f(x) \rightarrow f(y, y/\epsilon) = f(y, z) \).

\(^{77}\)The nomenclature used in eqns. (5.8) to (5.15) follows typical usage in asymptotic homogenization. It is more general than but can be directly compared to the one used in eqns. (1.2) to (5.7), where no macroscopic coordinates \( y \) are employed.
The first of these equations gives rise to a boundary value problem at the unit cell level that is referred to as the “micro equation”. By making a specific ansatz for the strains at the microlevel and by volume averaging over the second equation in the system (5.14), which is known as the “macro equation”, for elastic problems the microscopic and macroscopic fields can be linked such that the homogenized elasticity tensor, $E_{ijkl}^*$, is obtained as

$$E_{ijkl}^* = \frac{1}{\Omega_{UC}} \int_{\Omega_{UC}} E_{ijkl}(z) \left[ I_{klmn} + \frac{\partial}{\partial z_l} \chi_{kmn}(z) \right] d\Omega. \quad (5.15)$$

Here $\Omega_{UC}$ is the volume of the unit cell, $E_{ijkl}(z)$ is the microscopic elasticity tensor, which depends on the constituent present at position $z$, $I_{ijkl}$ is the 4th-order unit tensor, and the “characteristic function” $\chi_{kmn}(z)$, a tensor of order 3, describes the deformation modes of the unit cell and, accordingly, relates the micro- and macrofields. Analogous expressions can be derived for the tangent modulus tensors used in elastoplastic analysis, compare Ghosh et al. (1996).

The above relations can be used as the basis of Finite Element algorithms that solve for the characteristic function $\chi_{ijk}(z)$, a task that typically has required special analysis codes. For detailed discussions of asymptotic homogenization methods within the framework of FEM-based micromechanics see, e.g., Hollister et al. (1991), Ghosh et al. (1996), Hassani and Hinton (1999), Chung et al. (2001) or Kanouté et al. (2009). Asymptotic homogenization procedures for elastic composites using commercial FE packages were proposed, however, by Banks-Sills and Leiderman (1999), Barroqueiro et al. (2016) and Colera and Kim (2019), the latter employing a user defined element.

Asymptotic homogenization supports the direct coupling of FE models on the macro- and microscales, compare, e.g., Ghosh et al. (1996) or Terada et al. (2003), an approach that has been used in a number of multi-scale studies (compare chapter 8) and is sometimes referred to as the FE$^2$ method (Feyel, 2003). A treatment of homogenization in the vicinity of macroscopic boundaries can be found in Schrefler et al. (1997). Asymptotic homogenization schemes are suitable for handling finite strains and they have also been employed for problems combining higher-order stresses and strain gradients with nonlocal behavior on the macroscale (Kouznetsova et al., 2004), an approach referred to as higher order homogenization. For recent reviews of asymptotic homogenization see, e.g., Kalamkarov et al. (2009) and Charalambakis (2010).

An alternative unit-cell based asymptotic homogenization scheme solves for a displacement-like “fluctuation function” rather than for the characteristic function appearing in eqn. (5.15). This Variational Asymptotical Method for Unit Cell Homogenization (VA-MUCH) (Yu and Tang, 2007) also is suitable for implementation within a Finite Element framework. Developments of this approach are discussed, e.g., by Tang and Yu (2011).

---

78 Note that, even though eqns. (5.14) and (5.15) are derived from an explicit two-scale formulation, neither of them contains the scale parameter $\varepsilon$, see the discussion by Chung et al. (2001).

79 Such methods are especially useful for problems in which the length scales are not well separated; in them the “unit cells” do not necessarily remain periodic during the deformation process.
Method of Macroscopic Degrees of Freedom

When asymptotic homogenization is not used, it is good practice to apply far field stresses (in the case of load controlled analysis) or strains (in the case of displacement control) to a given unit cell via concentrated nodal forces or prescribed displacements, respectively, at the master nodes and/or pivot points. This approach was termed the “method of macroscopic degrees of freedom” by Michel et al. (1999).

For load controlled analysis, the nodal forces to be applied to the master nodes can be evaluated from the macroscopic stress $\sigma^a$ via the divergence theorem, see Smit et al. (1998). For the configuration shown in fig. 5.5 the concentrated forces acting on the master nodes SE and NW of a two-dimensional volume element, $f_{SE}$ and $f_{NW}$, can be shown to be given by the surface integrals

$$f_{SE} = \int_{\Gamma_E} t^a(s) \, d\Gamma \quad \text{and} \quad f_{NW} = \int_{\Gamma_N} t^a(s) \, d\Gamma . \tag{5.16}$$

Here $t^a(s) = \sigma^a * n_{\Gamma}(s)$ stands for the surface traction vector corresponding to the homogeneous macroscopic (applied, far field) stress field at some given point $s$ on the cell’s surface $\Gamma_{UC}$, and $n_{\Gamma}(s)$ is the local normal vector of the appropriate face. Equation (5.16) can be generalized to require that each master node is loaded by a force corresponding to the surface integral of the surface traction vectors over the face slaved to it via an equivalent of eqns. (5.16). Analogous procedures hold for three-dimensional cases, and symmetry as well as antisymmetry boundary conditions as described by eqns. (5.6) and (5.7), respectively, can be handled by eqn. (5.16).

For applying far field strains to periodic volume elements in displacement controlled analysis, the displacements to be prescribed to the master nodes must be obtained from the macroscopic strains via appropriate strain–displacement relations. For example, using the notation of eqns. (5.5), the displacement vectors to be prescribed to the master nodes NW and SE of the unit cell shown in fig. 5.5 can be evaluated from eqn. (5.3) as

$$u_{NW} = \varepsilon^a * c_W \quad \text{and} \quad u_{SE} = \varepsilon^a * c_S \tag{5.17}$$

for an applied strain $\varepsilon^a$ and linear strain–displacement relations. For appropriately chosen unit cells, the shift vectors $c_k$ are equal to the cell’s side lengths in the undeformed state and they are, accordingly, referred to as $c_W$ and $c_S$ in eqn. (5.17). In some cases displacement controlled unit cell models are somewhat easier to handle than load controlled ones.

---

80 Note that the $t^a(s)$ are not identical with the actual local values of the tractions at the cell boundaries, $t(s)$, which contain contributions due to the local field fluctuations. However, the $t^a(s)$ are equal to the $t(s)$ over a given cell face in an integral sense. For geometrically nonlinear analysis eqn. (5.16) must be applied to the current configuration. Because the far field stress $\sigma^a$ is constant it can be factored out of the surface integrals, so that the force applied to a master node $M$, $f_M$, can be expressed as $f_M = \sigma^a * \int_{\Gamma_M} n_{\Gamma}(z) \, d\Gamma$.

81 Equations (5.17) can be directly extended to three-dimensional configurations. For handling finite strains and geometrical nonlinearities within such a framework see, e.g., Huber et al. (2007) or Barulich et al. (2018).

82 Even though the loads acting on the master nodes obtained from eqn. (5.16) are equilibrated, in stress controlled analysis solid body rotations may be induced through small numerical errors. When these solid body rotations are suppressed by deactivating degrees of freedom beyond the ones required for enforcing periodicity, the quality of the solutions may be degraded or transformations of the averaged stresses and strains may be required.
In general, the overall stress and strain tensors within a unit cell can be evaluated by volume averaging\textsuperscript{83} or by using the equivalent surface integrals given in eqn. (1.4). In practice, it is often fairly straightforward to evaluate volume integrals by numerical integration schemes such as eqn. (4.8), whereas comparably convenient approximations are not available for surface integrals.

In the case of rectangular or hexahedral unit cells or reduced cells that are aligned with the coordinate axes, averaged engineering stress and strain components can, of course, be evaluated by dividing the applied or reaction forces at the master nodes by the appropriate surface areas and by dividing the displacements of the master nodes by the appropriate cell lengths, respectively. The displacements of and concentrated forces acting on the master nodes can also be used for evaluating the macroscopic stresses and strains from skewed unit cells having non-rectangular periodicity vectors, see, e.g., Pahr (2003), where also some additional aspects of unit cells, master nodes, and the method of macroscopic degrees of freedom are discussed. Care is required in applying as well as evaluating macroscopic stresses and strains if the volume elements are not quadrilaterals or hexahedra.

In order to obtain three-dimensional homogenized elastic tensors with the method of macroscopic degrees of freedom six suitable, linearly independent load cases must be solved for in the most general case.

5.4 Periodic Models for Composites Reinforced by Continuous Fibers

A wide range of periodic models have been reported for the most important groups of continuously reinforced composites, viz., unidirectionally reinforced, angle-ply and cross-ply materials, as well as fabric reinforced composites.

Composites Reinforced by Unidirectional Continuous Fibers

Composites reinforced by continuous, aligned fibers typically show a statistically transversely isotropic overall behavior and can be studied well with periodic homogenization. Materials characterization with the exception of the overall axial shear behavior can be carried out with two-dimensional unit cell or reduced cell models employing generalized plane strain elements that use a global degree of freedom for describing the axial deformation of the whole model\textsuperscript{84}. For handling the overall axial shear response (required, e.g., for establishing macroscopic elasticity tensors) special generalized plane strain elements (Adams and Crane, 1984; Sørensen, 1992) or three-dimensional models with appropriate

\textsuperscript{83}For finite strain analysis the consistent procedure is to first average the deformation gradients and then evaluate the strains from these averages.

\textsuperscript{84}Generalized plane strain elements suitable for such analysis are implemented in a number of commercial FE codes. Because the axial stiffness of composites reinforced by continuous aligned fibers can usually be satisfactorily described by Voigt-type (“rule of mixture”) models PMA studies of such materials have tended to concentrate on the transverse behavior. Note that plane strain models do not account for the axial constraints in such materials.
periodicity boundary conditions, see, e.g., Pettermann and Suresh (2000), are required. Three-dimensional modeling is required for materials characterization of composites reinforced by continuous aligned fibers when the effects of fiber misalignment, of fiber waviness (Garnich and Karami, 2004), of pores in the matrix, or of broken fibers (Mahishi, 1986) or of local damage to matrix or interface are to be studied. Unit cell models used for general nonlinear constitutive modeling of continuously reinforced composites must be fully three-dimensional and employ periodicity boundary conditions.

The most basic generalized plane strain models of continuously reinforced composites make use of simple periodic fiber arrangements as shown in fig. 5.8, all of which can be described by rather small reduced cells using symmetry and/or antisymmetry B.C.s, compare fig. 5.2. The simplest among these phase geometries are the periodic hexagonal (PH0) and periodic square (PS0) arrays, the use of which goes back to the 1960s (Adams and Doner, 1967). Models with hexagonal symmetry (PH0,CH1,RH2,CH3) give rise to transversely isotropic thermoelastic overall behavior whereas the other fiber arrangements shown in fig. 5.8 have tetragonal (PS0,CS7,CS8) or monoclinic (MS5) overall symmetry. In the elastoplastic and damage regimes the macroscopic symmetries of the fiber arrangements

\footnote{For linear elastic material behavior there is the additional option of describing out-of-plane shear behavior via the formal analogy between antiplane shear and diffusion (e.g., conduction) problems.}

\footnote{For investigating the axial failure behavior of continuously reinforced MMCs, statistical methods concentrating on fiber fragmentation within spring lattice models, see, e.g., Zhou and Curtin (1995), have been used successfully.}

\footnote{Like the macroscopic elastic responses, the phase averages of the local stress and strain fields of hexagonal arrangements of aligned fibers are independent of the orientations of transverse loads. The higher statistical moments and the distributions function of these fields at the phase level, however, do show such a dependence. Consequently, the equivalent stress varies to some extent with the loading direction, which underlies the small differences in initial yielding between the two “PH” curves in fig. 5.9.}
are degraded for most load cases, compare fig. 5.9, and hexagonal arrangements are not transversally isotropic with respect to ultimate strength. This behavior is due to the low symmetry of the distributions of the plastic strains in the matrix (and, as a consequence, the strain hardening state), which depend on the load history a given point has undergone. In many cases simple periodic microgeometries do not provide satisfactory descriptions of fiber reinforced materials, most of which show at least some randomness in the fiber positions. Much improved models can be obtained by periodic multi-fiber unit cells that employ quasi-random fiber positions. Such models can either use symmetry B.C.s, compare the cell shown in fig. 5.10, which is based on the work of Nakamura and Suresh (1993), or periodicity B.C.s (Moulinec and Suquet, 1997; Gusev et al., 2000; Zeman and Šejnoha, 2007). With growing computer power multi-fiber unit cells have become a standard tool for periodic homogenization of continuously reinforced composites.

![Figure 5.9: Transverse elastoplastic response of a unidirectional continuously reinforced ALTEX/Al MMC (\(\xi=0.453\), elastoplastic matrix with linear hardening) to transverse uniaxial loading as predicted by PMA models PH0, PS0 and DN.]

In table 5.1 thermoelastic moduli of an aligned continuously reinforced ALTEX/Al MMC as predicted by bounding methods, MFAs and unit cells methods using arrangements PH0, CH1 and PS0 (compare fig. 5.8) as well as DN (see fig. 5.10) are listed. For this material combination all PMA results (even for arrangement PS0, which is not transversally isotropic) fall within the Hashin–Shtrikman bounds\(^{88}\), but the predictions for the square arrangement show clear in-plane anisotropy and do not follow the three-point bounds (which in this case pertain to aligned, identical, non-overlapping cylindrical fibers, compare section 3.5). The results for the multi-fiber arrangement indicate some minor deviation from transversely isotropic macroscopic behavior.

\(^{88}\)The constituents’ material properties underlying table 5.1 show a low elastic contrast of \(c_{el} \approx 3\), making them relatively insensitive to perturbations of macroscopic symmetry. In general the elastic moduli obtained from square-type arrangements may violate the Hashin–Shtrikman bounds and usually lie outside the three-point bounds.
Table 5.1: Overall thermoelastic moduli of a unidirectional continuously reinforced ALTEX/Al MMC ($\xi = 0.453$ nominal) as predicted by the Hashin–Shtrikman (HSB) and three-point (3PB) bounds, by the Mori–Tanaka method (MTM), the generalized self-consistent scheme (GSCS), the differential scheme (DS) and Torquato’s three-point estimates (3PE), as well as by PMA analysis using periodic arrangements shown in fig. 5.8 (PH0, CH1, PS0) and the multi-fiber cell displayed in fig. 5.10 (DN). For arrangement PS0 responses in the 0° and 45°, and for arrangement DN responses in the 0° and 90° directions are listed.

<table>
<thead>
<tr>
<th></th>
<th>$E^*_A$ [GPa]</th>
<th>$E^*_T$ [GPa]</th>
<th>$\nu^*_A$</th>
<th>$\nu^*_T$</th>
<th>$\alpha^*_A$ [$K^{-1} \times 10^{-6}$]</th>
<th>$\alpha^*_T$ [$K^{-1} \times 10^{-6}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>fibers</td>
<td>180.0</td>
<td>180.0</td>
<td>0.20</td>
<td>0.20</td>
<td>6.0</td>
<td>6.0</td>
</tr>
<tr>
<td>matrix</td>
<td>67.2</td>
<td>67.2</td>
<td>0.35</td>
<td>0.35</td>
<td>23.0</td>
<td>23.0</td>
</tr>
<tr>
<td>HSB/lo 118.8</td>
<td>103.1</td>
<td>0.276</td>
<td>0.277</td>
<td>11.84</td>
<td>15.77</td>
<td></td>
</tr>
<tr>
<td>HSB/hi 119.3</td>
<td>107.1</td>
<td>0.279</td>
<td>0.394</td>
<td>12.47</td>
<td>16.46</td>
<td></td>
</tr>
<tr>
<td>3PB/lo 118.8</td>
<td>103.8</td>
<td>0.278</td>
<td>0.326</td>
<td>11.85</td>
<td>16.31</td>
<td></td>
</tr>
<tr>
<td>3PB/hi 118.9</td>
<td>104.5</td>
<td>0.279</td>
<td>0.347</td>
<td>11.98</td>
<td>16.45</td>
<td></td>
</tr>
<tr>
<td>MTM</td>
<td>118.8</td>
<td>103.1</td>
<td>0.279</td>
<td>0.342</td>
<td>11.84</td>
<td>16.46</td>
</tr>
<tr>
<td>GSCS</td>
<td>118.8</td>
<td>103.9</td>
<td>0.279</td>
<td>0.337</td>
<td>11.84</td>
<td>16.46</td>
</tr>
<tr>
<td>DS</td>
<td>118.8</td>
<td>103.9</td>
<td>0.278</td>
<td>0.339</td>
<td>11.94</td>
<td>16.35</td>
</tr>
<tr>
<td>3PE</td>
<td>118.8</td>
<td>103.9</td>
<td>0.279</td>
<td>0.338</td>
<td>11.89</td>
<td>16.40</td>
</tr>
<tr>
<td>PH0</td>
<td>118.8</td>
<td>103.7</td>
<td>0.279</td>
<td>0.340</td>
<td>11.84</td>
<td>16.46</td>
</tr>
<tr>
<td>CH1</td>
<td>118.7</td>
<td>103.9</td>
<td>0.279</td>
<td>0.338</td>
<td>11.90</td>
<td>16.42</td>
</tr>
<tr>
<td>PS0/00 118.8</td>
<td>107.6</td>
<td>0.279</td>
<td>0.314</td>
<td>11.85</td>
<td>16.45</td>
<td></td>
</tr>
<tr>
<td>PS0/45 118.8</td>
<td>99.9</td>
<td>0.279</td>
<td>0.363</td>
<td>11.85</td>
<td>16.45</td>
<td></td>
</tr>
<tr>
<td>DN/00 118.8</td>
<td>104.8</td>
<td>0.278</td>
<td>0.334</td>
<td>11.90</td>
<td>16.31</td>
<td></td>
</tr>
<tr>
<td>DN/90 118.8</td>
<td>104.6</td>
<td>0.278</td>
<td>0.333</td>
<td>11.90</td>
<td>16.46</td>
<td></td>
</tr>
</tbody>
</table>

The fiber arrangements shown in figs. 5.8 and 5.10 give nearly identical results for the overall thermoelastoplastic behavior of continuously reinforced composites under axial mechanical loading, and the predicted overall axial and transverse responses under thermal loading are also very similar. The overall behavior under transverse mechanical loading, however, depends markedly on the phase arrangement, see fig. 5.9. For fiber arrangements of tetragonal or lower symmetry (e.g., PS0, MS5, CS7 and CS8) the predicted transverse stress-strain responses depend strongly on the loading direction. The behavior of the hexagonal arrangements is sandwiched between the stiff (0°) and the compliant (45°) responses of periodic square arrangements in both the elastic and elastoplastic ranges. Multi-fiber cells with statistical fiber positions (such as the one shown in fig. 5.10) tend to show noticeably stronger macroscopic strain hardening compared to periodic hexagonal arrangements of the same fiber volume fraction, compare Moulinec and Suquet (1997). Typically, the macroscopic yield surfaces of uniaxially reinforced MMCs are not of the Hill (1948) type, but can be described by a bimodal description (Dvorak and Bahei-el Din, 1987). Analogous macroscopic behavior has been reported for viscoelastic composites (Li et al., 2015).

The distributions of microstresses and microstrains in fibers and matrix typically depend strongly on the fiber arrangement, especially under thermal and transverse mechanical
Figure 5.10: Microscopic distributions of the accumulated equivalent plastic strain in the matrix of a transversely loaded unidirectional continuously reinforced ALTEX/Al MMC (ξ=0.453) as predicted by a multi-fiber cell (arrangement DN).

loading. In the plastic regime, the microscopic distributions of equivalent and hydrostatic stresses, equivalent plastic strains and stress triaxialities tend to be markedly inhomogeneous\textsuperscript{89}, see, e.g., fig. 5.10. As a consequence of the inhomogeneity of the microfields, there tend to be strong constrained plasticity effects in continuously reinforced MMCs and the onset of damage in the matrix, of fracture of the fibers, and of interfacial decohesion at the fiber–matrix interfaces show a strong dependence on the fiber arrangement.

Cross-Ply and Angle-Ply Composites

Another group of composite materials reinforced by continuous fibers that can be studied to advantage by unit cell methods are laminates consisting of plies the thickness of which is not much greater than the fiber diameter. The left side of fig. 5.11 depicts a unit cell for a cross-ply laminate with double layers of fibers which is suitable for use with periodicity and symmetry boundary conditions. In the the center a minimum reduced cell for cross ply laminates with one fiber layer per ply is shown, which requires the use of symmetry boundary conditions. A unit cell for studying angle-ply laminates with a general ply angle β via periodicity boundary conditions is displayed on the right side of fig. 5.11. Extending models of the above type to multi-fiber cells with randomly arranged fiber positions appears feasible.

\textsuperscript{89} The corresponding distribution functions, phase averages, and higher statistical moments are also significantly influenced by the fiber arrangement, compare Böhm and Rammerstorfer (1995).
Figure 5.11: Unit cell for a double layer cross-ply laminate (left), minimum reduced cell for a cross-ply-laminate (center) and unit cell for an angle-ply laminate with ply angle $\beta$ (right).

Unit cells with symmetry boundary conditions were used for studying the thermomechanical behavior of cross ply laminates, e.g., by Lerch et al. (1991) as well as Ismar and Schröter (2000), and volume elements with periodicity BC, e.g., by Soni et al. (2014). For unit cell studies of angle ply laminates see, e.g., Xia et al. (2003) and Abolfathi et al. (2008).

**Fabric Reinforced Composites**

Periodic homogenization has played an important role in modeling the behavior of fabric reinforced composites, i.e., materials containing woven, braided, or knitted reinforcements. In such “textile composites” the reinforcing phase takes the form of textile-like structures consisting of interlacing bundles of continuous fibers (tows). Unit cell and reduced cell models for such materials are typically based on modeling fiber bundles as a “mesophase” with smeared-out material properties, which, in turn, are obtained from analyzing uni-directionally continuously reinforced composites. Free surface boundary conditions are specified for the out-of-plane faces of the cell, and a number of cells may be stacked on top of each other in order to account for in-plane offsets and constraint effects between the layers (Bystöm et al., 2000). Symmetry boundary conditions can be specified for the in-plane faces, giving reasonably small unit cells as shown in fig. 5.12, and combinations of periodicity, symmetry and antisymmetry boundary conditions may be used to achieve very small volume elements, see, e.g., (Tang and Whitcomb, 2003). Such models, however, are restricted to handling in-plane normal and thermal loads. By applying extended periodicity boundary conditions to larger unit cells macroscopic rotational degrees of freedom can be introduced to allow studying all deformation modes, including the macroscopic warping and twisting of the laminae.

There is a wide range of weaves as well as knitting and braiding architectures that can be modeled with PMAs. Unit cells for woven, knitted and braided composites tend to be fairly complex geometrically, may be difficult to mesh for Finite Element analysis and may pose considerable computational requirements, especially when nonlinear behavior is to be studied. Over the past 25 years a considerable number of studies have been published on unit cell modeling of fabric reinforced composites, see, e.g., Cox and Flanagan (1997),

---

90: This modeling strategy obviously is a type of multiscale modeling as discussed in chapter 8.
Huang and Ramakrishna (2000), Tang and Whitcomb (2003) or Lomov et al. (2006), and on software for generating appropriate unit cells (Robitaille et al., 2003; Sherburn, 2007). Gager and Pettermann (2012) proposed shell element-based unit cells for textile composites that can significantly reduce the modeling effort.

5.5 Periodic Models for Composites Reinforced by Short Fibers

The overall symmetry of short fiber reinforced composites in many cases is isotropic (for random fiber orientations) or transversely isotropic (for aligned fibers, planar random fibers and other fiber arrangements with axisymmetric orientation distributions). However, processing conditions can give rise to a wide range of fiber orientation distributions and, consequently, lower overall symmetries (Allen and Lee, 1990). The thermoelastic and thermoelastoplastic behavior of aligned short fiber reinforced composites has been successfully estimated by Mori–Tanaka methods\(^\text{91}\), which can also be extended to nonaligned fibers and reinforcements showing an aspect ratio distribution, compare section 2.6. Such mean-field approaches are, however, limited in resolving details of fiber arrangements, especially for inelastic material behavior. At present the most powerful tools for studying the influence of fiber shapes and orientations, of clustering effects, of the interaction of fibers of different sizes, and of local stress and strain fields between neighboring fibers are periodic microfield methods. Platelet reinforced composites can be described by analogy to short fiber reinforced materials.

\(^{91}\)For a comparison between unit cell and analytical predictions for the overall elastic properties of short fiber reinforced composites see, e.g., Tucker and Liang (1999).
Composites Reinforced by Aligned Short Fibers

In contrast to continuously reinforced composites, the phase arrangements of discontinuously reinforced materials are inherently three-dimensional. The simplest three-dimensional PMA models of aligned short fiber reinforced composites have used periodic square arrangements of non-staggered or staggered aligned cylindrical fibers see, e.g., Levy and Papazian (1991) and compare fig. 5.13. Such geometries are relatively simple to set up and do not pose major computational requirements, but are rather restrictive in terms of fiber arrangements and load conditions that can be handled. By using larger volume elements supporting periodicity boundary conditions the full thermomechanical behavior of the fiber arrangements can be studied. Analogous microgeometries based on periodic hexagonal arrangements of non-staggered (Järvsträt, 1992) or staggered (Tucker and Liang, 1999) fibers as well as ellipsoidal cells aimed at describing composite “ellipsoid assembly” microgeometries (Järvsträt, 1993) were also proposed.

Figure 5.13: Three-dimensional cells for modeling non-staggered (left) and staggered (right) square arrangements of aligned short fibers. The shaded reduced cells, which follow Levy and Papazian (1991), require symmetry B.C., cells outlined in bold are suitable for periodicity B.C.

For many materials characterization studies, a more economical alternative to the above three-dimensional unit cells takes the form of axisymmetric models describing the axial behavior of non-staggered or staggered arrays of aligned cylindrical short fibers in an approximate way. The basic idea behind these models is replacing unit cells for square or hexagonal arrangements by circular composite cylinders of equivalent cross sectional area (and volume fraction) as sketched in Fig. 5.14. The resulting axisymmetric cells are not

\[92\] Such square arrangements give rise to tetragonal overall symmetry, and, consequently, the transverse overall properties are direction dependent.

109
proper unit cells, because they overlap and are not space filling. In addition, their association with three-dimensional phase arrangements is somewhat tenuous — note that they do not show the same transverse fiber spacing as the “equivalent” periodic arrangements in Fig. 5.14 — and they are severely limited in the type of loading conditions they can handle. However, they have the advantage of significantly reduced computational requirements, which has made them especially suitable for qualitative studies of highly nonlinear behavior such as damage.

Symmetry boundary conditions are used for the top and bottom faces of the cells shown in Fig. 5.14. The circumferential surfaces must be chosen to enforce identical cross-sectional areas along the axial direction for any aggregate of cells. In the case of non-staggered fibers this can be easily done by specifying symmetry-type boundary conditions, eqns. (5.6), for the outer surfaces.

For staggered arrangements a pair of cells with different fiber positions is considered, the total cross-sectional area of which is required to be independent of the axial coordinate. By choosing the two cells making up the pair such that they are antisymmetric with respect to a pivot point P the behavior of the staggered arrangement can be described by a single cell with an antisymmetric outer (E-) face, U and L being nodes on this face that are positioned symmetrically with respect to the pivot. Using the nomenclature of fig. 5.15

---

93 Axisymmetric cells are limited to load cases in which the deformed shape of the cell remains axisymmetric. These include uniaxial stress and strain, volumetric stress, “in plane hydrostatic stress” (where a homogeneous radial stress is prescribed) and isochoric strain (obtained, in the small strain regime, by applying constant normal displacements at the “upper” and “outer” faces of, \( u_N = 2u_E \), respectively).

94 As originally proposed by Tvergaard (1990) the nonlinear displacement boundary conditions in eqn. (5.18) were combined with antisymmetry traction B.C.s for use with a hybrid FE formulation.
and the notation of eqns. (5.5) to (5.7), this leads to nonlinear relations for the radial displacements \( u \) and linear constraints for the axial displacements \( v \) at the outer surface,

\[
(r_U + u_U)^2 + (r_L + u_L)^2 = 2(r_P + u_P)^2 \quad \text{and} \quad v_U + v_L = 2v_P ,
\]

respectively, where \( r \) is the radius of the undeformed cell. Nonlinear constraints such as eqn. (5.18), however, tend to be cumbersome to use and are not widely available in FE codes\(^{95}\). For typical small-strain problems the boundary conditions for the radial displacements at the outer surfaces of the cylinders, eqn. (5.18), can be linearized without major loss in accuracy to give sets of linear constraint equations

\[
(u_U + u_L = 2u_P \quad \text{and} \quad v_U + v_L = 2v_P ) ,
\]

which can be seen to be formally identical to the antisymmetry boundary conditions described by eqn. (5.7).

Axisymmetric cell models of the types shown in fig. 5.14 were the the workhorses of PMA studies of short fiber reinforced composites, in the 1990s, see, e.g., Povirk et al. (1992) or Tvergaard (1994). Typically, descriptions using staggered arrangements allow a wider range of microgeometries to be covered, compare Böhm et al. (1993) or Tvergaard (2003), and give more realistic descriptions of actual composites. Both staggered and non-staggered axisymmetric models can be extended to studying a considerable range of arrangements incorporating aligned fibers of different size and/or aspect ratio by coupling two or more different cells via the condition of keeping the cross sectional area of the aggregate independent of the axial coordinate (Böhm et al., 1993).

Axisymmetric cells and simple three-dimensional unit cells have been used successfully for studying the nonlinear thermomechanical behavior of aligned short fiber reinforced MMCs, e.g., with respect to their stress–strain responses, to the pseudo-Bauschinger

\(^{95}\)For an example of the use of the nonlinear B.C.s described in eqn. (5.18) with a commercial FE code see Ishikawa et al. (2000), where cells of truncated cone shape are employed to describe bcc arrangements of particles.
effect, and to thermal residual stresses. They have provided valuable insight into causes and effects of matrix, interface and fiber damage. Over the past years, however, multi-fiber unit cells have become the standard tool for studying composites reinforced by aligned short fibers.

Composites Reinforced by Nonaligned Short Fibers

The first unit cell studies of materials reinforced by aligned discontinuous fibers were based on three-dimensional models of composites reinforced by alternatingly tilted misaligned fibers (Sørensen et al., 1995) and plane-stress models describing planar random short fibers (Courage and Schreurs, 1992). Multi-fiber unit cells of such materials have started coming into their own in the 2000s. It is worth noting that the situation with respect to mean-field models for composites reinforced by short fibers is not fully satisfactory for the elastic range, compare section 2.6, and limited work has been done on their inelastic thermomechanical behavior.

At present the most powerful continuum modeling strategy for nonaligned short fiber reinforced composites consists of using three-dimensional multi-fiber volume elements in which the fibers are randomly positioned and oriented such that the required ODF is fulfilled to a suitable level of approximation. If the geometries are periodic, they are obviously suitable for periodic homogenization. Among early reports for such multi-fiber unit cells were, e.g., Lusti et al. (2002) or Böhm et al. (2002) for spatially random fiber orientations and Duschlbauer et al. (2006) or Iorga et al. (2008) for planar random fiber orientations. Models of this type can also account for the distributions of fiber sizes and/or aspect ratios. Such modeling approaches tend to pose considerable challenges in generating appropriate fiber arrangements at non-dilute volume fractions due to geometrical frustration, relatively large volume elements being required for handling periodic microgeometries with fiber aspect ratio in excess of, say, 5. The meshing of the resulting phase geometries with fiber aspect ratio in excess of, say, 5. The meshing of the resulting phase arrangements for use with the FEM may also be difficult, compare Shephard et al. (1995), and analyzing the mechanical response of the resulting cells may require considerable computing power, especially for nonlinear constituent behavior. The first studies of this type were, accordingly, restricted to the linear elastic range, where the BEM has been found to answer well, see, e.g., Banerjee and Henry (1992). Commercial micromechanics codes such as DIGIMAT (e-Xstream Engineering, Mont-Saint-Guibert, Belgium) or PALMYRA (MatSim Ges.m.b.H., Zurich, Switzerland) have become available that provide capabilities for setting up and solving periodic models with considerable numbers of nonaligned short fibers.

As a simple example of a multi-fiber unit cell for a composite reinforced by nonaligned short fibers, fig. 5.16 shows a cube-shaped cell that contains 15 randomly oriented cylindrical fibers of aspect ratio \( a = 5 \), supports periodicity boundary conditions, was generated by random sequential addition, and is discretized by tetrahedral elements. The phase arrangement is set up in such a way that spheroidal fibers of the same aspect ratio, volume fractions, center points, and orientations can also be used in order to allow studying fiber shape effects (Böhm et al., 2002).

Table 5.2 lists predictions for the overall elastic response of a composite reinforced by randomly positioned and oriented fibers obtained by analytical estimates, the Hashin–
Figure 5.16: Unit cell for a composite reinforced by randomly oriented short fibers (Böhm et al., 2002). The nominal fiber volume fraction is $\xi = 0.15$ and the 15 cylindrical fibers in the cell have an aspect ratio of $a = 5$.

Shtrikman bounds (Hashin and Shtrikman, 1963) and results obtained with the above type of unit cell, for which data are shown pertaining to cylindrical and spheroidal fibers. Interestingly, noticeable differences were found between the predictions for reinforcement by cylindrical and spheroidal fibers that have the same fiber volume fraction and aspect ratio, occupy the same positions and show the same orientations, with the spheroidal rein-

Table 5.2: Overall elastic properties of a SiC/Al2618 MMC reinforced by randomly oriented fibers ($a = 5$, $\xi = 0.15$) as predicted by the Hashin–Shtrikman bounds (HSB), by the extended Mori–Tanaka method (MTM) of Wei and Edwards (1999), by the classical self-consistent scheme (CSCS) of Berryman (1980), by the Kuster and Toksöz (1974) model (KTM) and by multi-fiber unit cells of the type shown in fig. 5.16 (MFUC), which contain 15 spheroidal or cylindrical fibers (Böhm et al., 2002).

<table>
<thead>
<tr>
<th></th>
<th>$E^*$</th>
<th>$\nu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>fibers</td>
<td>450.0</td>
<td>0.17</td>
</tr>
<tr>
<td>matrix</td>
<td>70.0</td>
<td>0.30</td>
</tr>
<tr>
<td>HSB/lo</td>
<td>87.6</td>
<td>0.246</td>
</tr>
<tr>
<td>HSB/hi</td>
<td>106.1</td>
<td>0.305</td>
</tr>
<tr>
<td>MTM</td>
<td>89.8</td>
<td>0.285</td>
</tr>
<tr>
<td>CSCS</td>
<td>91.2</td>
<td>0.284</td>
</tr>
<tr>
<td>KTM</td>
<td>90.3</td>
<td>0.285</td>
</tr>
<tr>
<td>MFUC/sph</td>
<td>89.4</td>
<td>0.285</td>
</tr>
<tr>
<td>MFUC/cyl</td>
<td>90.0</td>
<td>0.284</td>
</tr>
</tbody>
</table>
forcements leading to a more compliant response, especially in the inelastic regime. Useful results were obtained despite the low number of fibers in the unit cell and excellent agreement with analytical descriptions was achieved in terms of the orientation dependence of the stresses in individual fibers, compare fig. 2.4. Whereas the phase arrangement shown in fig. 5.16 does not contain a sufficient number of fibers for approaching representativeness even in the elastic range, recent studies using much larger volume elements have markedly improved this situation, see, e.g., Tian et al. (2015).

5.6 Periodic Models for Particle Reinforced Composites

Actual particle reinforced composite materials often show rather irregular particle shapes, and anisotropy in the microgeometry as well as in the overall responses may be introduced by processing effects such as extrusion textures. Nevertheless, for materials reinforced by statistically uniformly distributed equiaxed particles isotropic macroscopic behavior is a useful approximation, applied in many modeling studies. The thermoelastic behavior of macroscopically isotropic composites has been successfully described by mean-field methods that approximate the particles as spheres, see section 2.3. Extensions of the mean-field solutions into the nonlinear range are available, compare section 2.5, but they are subject to some limitations in predicting the overall thermomechanical response in the post-yield regime. In addition, mean-field models are limited in accounting for many particle shape, clustering, and size distribution effects and cannot resolve local fluctuations of the stress and strain fields. As a consequence, the past 30 years have witnessed marked interest in periodic homogenization models for studying the thermomechanical behavior of particle reinforced composites.

One issue in applying periodic microfield approaches to the modeling of particle reinforced materials with macroscopically isotropic behavior is that there exists no simple periodic three-dimensional phase arrangement that shows matrix–inclusion topology and is inherently elastically isotropic\(^\text{96}\). Together with the wide variation in microgeometries and particle shapes in actual materials, this gives rise to nontrivial tradeoffs between keeping computational requirements at reasonably low levels (favoring simple particle shapes combined with two-dimensional or simple three-dimensional microgeometries) and obtaining sufficiently realistic models for a given purpose (typically best fulfilled by three-dimensional volume elements containing considerable numbers of randomly positioned particles of complex shape). In many respects periodic models of particle reinforced composites are subject to similar constraints and use analogous approaches as work on short fiber reinforced composites.

Many three-dimensional unit cell studies of generic microgeometries for particle reinforced composites have been based on simple cubic (sc), face centered cubic (fcc) or body centered cubic (bcc) arrangements of spherical, cylindrical or cube-shaped particles, com-

\(^{96}\)Even though volume elements in the shape of pentagonal dodecahedra and icosahedra have the appropriate symmetry properties (Christensen, 1987), they are not space filling.
Figure 5.17: Simple cubic, face centered cubic, and body centered cubic arrangements of particles plus some pertinent small reduced cells (Weissenbek et al., 1994).

Compare, e.g., Hom and McMeeking (1991). By invoking the symmetries of these arrangements and using symmetry as well as antisymmetry boundary conditions, small reduced cells for materials characterization can be obtained\(^{97}\), compare Weissenbek et al. (1994) and see figs. 5.17 and 5.18. In addition, work employing hexagonal or tetrakaidecahedral particle arrangements (Rodin, 1993) and Voronoi cells for cubic arrangements of particles (Li and Wongsto, 2004) was reported.

Figure 5.18: Some reduced cells for particle reinforced composites using cubic arrangements of inhomogeneities: s.c. arrangement of cubes, b.c.c. arrangement of cylinders and f.c.c. arrangement of spheres (Weissenbek et al., 1994).

The effective elastic moduli obtained from cubic arrangements of particles show cubic symmetry\(^{98}\), do not necessarily fulfill the Hashin–Shtrikman bounds for isotropic materials, and typically lie outside the three-point bounds, compare table 5.3. Of the phase arrangements shown in fig. 5.17, simple cubic models are the easiest to handle, but they show a more marked anisotropy than bcc and fcc ones.

\(^{97}\) Among the reduced cells shown in fig. 5.18 the one on the left uses symmetry boundary conditions and is restricted in the load cases that can be handled. The other two cells employ antisymmetry B.C.s and can be used more freely. Unit cells suitable for periodicity boundary conditions are considerably larger, but allow unrestricted modeling of the full thermomechanical response of cubic phase arrangements.

\(^{98}\) Cubic symmetry is a special case of orthotropy with equal responses in all principal directions and requires three independent moduli for describing the elastic behavior. For a discussion of the elastic anisotropy of cubic materials see, e.g., Cazzani and Rovati (2003). The conduction and diffusion behavior of materials with cubic symmetry is isotropic.
During the past 20 years studies based on increasingly complex three-dimensional phase arrangements have assumed a prominent role in the literature. Gusev (1997) used Finite Element methods in combination with unit cells containing up to 64 statistically positioned particles to describe the overall behavior of elastic particle reinforced composites. Hexahedral unit cells containing up to 10 particles in a perturbed cubic configuration (Watt et al., 1996) as well as cube shaped cells incorporating at least 15 spherical particles in quasi-random arrangements (Böhm et al., 1999; Böhm and Han, 2001), compare fig. 5.19, or clusters of particles (Segurado et al., 2003; Lee et al., 2011) were used in Finite Element based studies of elastoplastic particle reinforced MMCs and related materials. Three-dimensional simulations involving high numbers of particles have been reported, e.g., for investigating the elastic behavior of composites (Michel et al., 1999), for studying brittle matrix composites that develop damage (Zohdi and Wriggers, 2001), and for rubber reinforced polymers (Fond et al., 2001).

Figure 5.19: Unit cell for a particle reinforced MMC ($\xi=0.2$) containing 20 spherical particles in a quasi-random arrangement that is suitable for using periodicity B.C. (Böhm and Han, 2001).

Axisymmetric cells describing staggered or non-staggered arrangements of particles can be used for materials characterization of particle reinforced composites in full analogy with short fiber reinforced materials, compare figs. 5.14 and 5.15. By appropriate choice of the dimensions of the axisymmetric cells sc, fcc and bcc arrangements of particles can be approximated to some extent, and a range of axisymmetric particle shapes can be studied. Such models were a mainstay of PMA modeling of materials containing particulate inhomogeneities, see, e.g., Bao et al. (1991) or LLorca (1996). A related type of model are spherical cells (Guild and Kinloch, 1995).

Due to their low computational requirements, planar unit cell models of particle reinforced materials have also been used to a considerable extent. Typically, plane stress
Table 5.3: Overall thermoelastic properties of a particle reinforced SiC/Al MMC (spherical particles, $\xi=0.2$) predicted by the Hashin–Shtrikman (HSB) and three-point (3PB) bounds, by the Mori–Tanaka method (MTM), the generalized self-consistent scheme (GSCS), the differential scheme (DS), and Torquato’s three-point estimates (3PE), as well as by unit cell analysis using three-dimensional cubic arrangements, axisymmetric cells approximating sc, fcc and bcc geometries, two-dimensional multi-particle models based on plane stress (2/D PST) and plane strain (2/D PSE) kinematics, as well as three-dimensional multi-particle models as shown in fig. 5.19.

<table>
<thead>
<tr>
<th></th>
<th>$E^{*}$</th>
<th>$E^{*}[100]$</th>
<th>$E^{*}[110]$</th>
<th>$\nu^{*}$</th>
<th>$\alpha^{\lambda}_{\Lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>particles</td>
<td>429.0</td>
<td>—</td>
<td>—</td>
<td>0.17</td>
<td>4.3</td>
</tr>
<tr>
<td>matrix</td>
<td>67.2</td>
<td>—</td>
<td>—</td>
<td>0.35</td>
<td>23.0</td>
</tr>
<tr>
<td>HSB/lo</td>
<td>90.8</td>
<td>—</td>
<td>—</td>
<td>0.286</td>
<td>16.8</td>
</tr>
<tr>
<td>HSB/hi</td>
<td>114.6</td>
<td>—</td>
<td>—</td>
<td>0.340</td>
<td>18.6</td>
</tr>
<tr>
<td>3PB/lo</td>
<td>91.4</td>
<td>—</td>
<td>—</td>
<td>0.323</td>
<td>18.5</td>
</tr>
<tr>
<td>3PB/hi</td>
<td>93.9</td>
<td>—</td>
<td>—</td>
<td>0.328</td>
<td>18.6</td>
</tr>
<tr>
<td>MTM</td>
<td>90.8</td>
<td>—</td>
<td>—</td>
<td>0.329</td>
<td>18.6</td>
</tr>
<tr>
<td>GSCS</td>
<td>91.5</td>
<td>—</td>
<td>—</td>
<td>0.327</td>
<td>18.6</td>
</tr>
<tr>
<td>DS</td>
<td>92.7</td>
<td>—</td>
<td>—</td>
<td>0.326</td>
<td>18.4</td>
</tr>
<tr>
<td>3PE</td>
<td>91.8</td>
<td>—</td>
<td>—</td>
<td>0.327</td>
<td>18.6</td>
</tr>
<tr>
<td>sc</td>
<td>—</td>
<td>96.4</td>
<td>90.5</td>
<td>—</td>
<td>18.7</td>
</tr>
<tr>
<td>fcc</td>
<td>—</td>
<td>89.0</td>
<td>91.7</td>
<td>—</td>
<td>18.6</td>
</tr>
<tr>
<td>bcc</td>
<td>—</td>
<td>90.0</td>
<td>90.6</td>
<td>—</td>
<td>18.6</td>
</tr>
<tr>
<td>axi/sc</td>
<td>95.4</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>18.6/18.6</td>
</tr>
<tr>
<td>axi/fcc</td>
<td>88.1</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>18.1/19.0</td>
</tr>
<tr>
<td>axi/bcc</td>
<td>87.9</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>18.4/18.8</td>
</tr>
<tr>
<td>2/D PST</td>
<td>85.5</td>
<td>—</td>
<td>—</td>
<td>0.334</td>
<td>—</td>
</tr>
<tr>
<td>2/D PSE</td>
<td>98.7</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>3/D</td>
<td>92.4</td>
<td>—</td>
<td>—</td>
<td>0.326</td>
<td>—</td>
</tr>
</tbody>
</table>

models (which actually describe thin “reinforced sheets” or the stress states at the surface of inhomogeneous bodies) show a more compliant and plane strain models (which describe reinforcing by aligned continuous fibers rather than by particles) show a stiffer overall response than three-dimensional descriptions, compare table 5.3. With respect to the overall behavior, plane stress analysis may be preferable to plane strain analysis, compare (Weissenbek, 1994), but no two-dimensional model gives satisfactory results in terms of the predicted microstress and microstrain distributions⁹⁹ (Böhms and Han, 2001).

⁹⁹One issue giving rise to difficulties in describing macroscopically isotropic phase arrangements with two-dimensional models is the lack of a proper hydrostatic load case. Its closest planar equivalent, equi-biaxial loading, is controlled by the plane strain bulk modulus, $K_T$, or the plane stress bulk modulus, respectively, both of which differ from the “three-dimensional” bulk modulus $K$.

In plastic regimes, plane stress configurations tend to give much higher equivalent plastic strains in the matrix and weaker overall hardening than do plane strain and generalized plane strain models using the same planar phase geometry. The configurations of regions of concentrated strains that underlie this behavior depend strongly on the geometrical constraints, compare, e.g., Iung and Grange (1995), Gänsler et al. (1998), Böhms et al. (1999) or Shen and Lissenden (2002). For continuously reinforced composites under transverse loading the equivalent plastic strains tend to concentrate in bands oriented at 45° to the loading directions, whereas the patterns are qualitatively different in particle reinforced materials.

117
Axisymmetric cell models typically provide considerably better results for the behavior of particle reinforced composites than do planar ones.

In Table 5.3 bounds, MFA results and a number of PMA predictions for the overall thermoelastic moduli of a particle reinforced SiC/Al MMC (elastic contrast $c_{el} \approx 5.5$) are compared, the loading directions for the cubic arrangements being identified by Miller indices. It is noteworthy that — for the loading directions considered here, which do not span the full range of possible responses of the cells — none of the cubic arrangements gives results that fall within the three-point bounds for identical spherical particles\(^{100}\). The overall anisotropy of the simple cubic arrangement is marked, whereas the body centered and face centered arrays deviate much less from isotropy. The predictions of the axisymmetric models can be seen to be of comparable quality in terms of the elastic moduli to those of the corresponding cubic arrays, but typically give a more anisotropic thermal expansion behavior. The results listed for the three-dimensional multi-particle models are ensemble averages over a number of unit cells and loading directions (compare the discussion in Section 4.1), and they show very good agreement with the three-point bounds and estimates, whereas there are marked differences to the plane stress and plane strain models. For the composite studied here, predicted overall moduli evaluated from the individual multi-particle cells differ by about 2%, and due to the use of a periodic pseudo-random rather than a statistically homogeneous particle distribution, the estimates of Drugan and Willis (1996) for the errors in the overall moduli are exceeded to some extent.

![Figure 5.20: Predicted distribution of equivalent plastic strain in the matrix of a particle reinforced MMC ($\xi=0.2$) subjected to uniaxial tensile loading (in left–right direction) obtained for a unit cell with 20 spherical particles in a quasi-random arrangement (Böhm and Han, 2001).](image)

Similarly to fiber reinforced MMCs, particle reinforced composites typically display

\(^{100}\)For cubic symmetries the extremal values of the Young’s modulus are found for loading in the [100] and [111] directions (Nye, 1957).
highly inhomogeneous microscopic stress and strain distributions, especially in nonlinear regimes. As an example, fig. 5.20, shows the predicted equivalent plastic strains of the elastoplastic matrix inside the multi-particle unit cell model depicted in fig. 5.19. This behavior tends to give rise to microscopic “structures” that can be considerably larger than individual particles, leading to longer ranged interactions between inhomogeneities. The latter, in turn, underlie the need for larger volume elements for studying composites with nonlinear matrix behavior mentioned in chapter 4.

Periodic volume elements are highly suitable for carrying out numerical experiments that explore the influence of different aspects of the microgeometry of particle reinforced composites on their microscopic and macroscopic responses. For example, Rasool and Böhm (2012) and Böhm and Rasool (2016) studied model composites that contain equal volume fractions of randomly positioned and, where applicable, randomly oriented, identical particles having the shapes of spheres, cubes, regular octahedra and regular tetrahedra, respectively. Figure 5.21 shows the resulting predictions for the responses to a single, non-symmetric, uniaxial stress load cycle, clear effects of the particle shape on the hardening behavior and on the residual strains being evident.

Figure 5.21: PMA predictions for the responses to a uniaxial load cycle of elastoplastic, particle reinforced MMCs ($\xi=0.2$) containing identical spherical (SPH), cube-shaped (CUB), octahedral (OCT) and tetrahedral (TET) inhomogeneities, respectively.

If the particles are highly irregular in shape or if their volume fraction markedly exceeds 0.5 (as is typically the case for cermets such as WC/Co), generic unit cell models employing relatively regular particle shapes may not result in satisfactory microgeometries. For such materials a useful approach consists of basing models on (modified) real structures obtained from metallographic sections, compare, e.g., Fischmeister and Karlsson (1977). Due to the

---

101 Due to the filtering effect of periodic phase arrangements mentioned in section 5.1, the length of such features can exceed the size of the unit cell only in certain directions, which may induce a spurious direction dependence into the predicted large-strain behavior of sub-RVE unit cells.
nature of the underlying experimental data, models of this type have often taken the form of planar analysis, the limitations of which are discussed above. Obviously, this modeling strategy can be improved by using three-dimensional data obtained, e.g., from digitized serial sections, compare Terada and Kikuchi (1996), or microtomographic scans, see, e.g., Borbély et al. (2006).

5.7 Periodic Models for Porous and Cellular Materials

Due to their relevance to the ductile damage and failure of metallic materials elastoplastic porous materials have been the subjects of a considerable number of PMA studies\textsuperscript{102}. Generally, modeling concepts for porous materials are closely related to the ones employed for particle reinforced composites, the main difference being that the shapes of the voids may evolve significantly through the loading history\textsuperscript{103}. Axisymmetric cells of the types discussed in sections 5.5 and 5.6, compare, e.g., Koplik and Needleman (1988) or Gărăjeu et al. (2000), and three-dimensional unit cells based on cubic arrangements of voids, see, e.g., McMeeking and Hom (1990) or Segurado et al. (2002b), have been used in the majority of pertinent PMA studies. In addition, studies of void growth in ductile materials based on multi-void cells and using geometry data from serial sectioning were reported (Shan and Gokhale, 2001).

In cellular materials, such as foams, wood and trabecular bone, the volume fraction of the solid phase is very low (often amounting to no more than a few percent) and the void phase may be topologically connected (open cell foams), unconnected (closed cell foams, syntactic foams), or both of the above (e.g., hollow sphere foams). The linear elastic regime of cellular materials in many cases is limited to a very small range of macroscopic strains. Furthermore, marked differences tend to be present between tensile and compressive inelastic macroscopic responses: As strains increase, gross shape changes of the cells typically take place on the microscale, with large bending deformations, the formation of plastic hinges, elastic as well as plastic buckling, and brittle failure of struts and cells walls playing major roles, especially under compressive loading. For compression-dominated load cases this regime tends to give rise to a stress plateau on the macroscale, which underlies the favorable energy absorption properties of many cellular materials. At some elevated strain the effective stiffness under compression typically rises sharply, the cellular structure

\textsuperscript{102}Many models and constitutive descriptions of the ductile damage and failure of metals, among them contributions by Rice and Tracey (1969), Gurson (1977), Tvergaard and Needleman (1984), Gologanu et al. (1997) as well as Benzerga and Besson (2001), are based on micromechanical studies of the growth of pre-existing voids in a ductile matrix.

\textsuperscript{103}The evolution of the shapes of initially spheroidal voids under non-hydrostatic loads has been the subject of studies by mean-field type methods, compare Kailasam et al. (2000). Such models use the assumption that initially spherical pores will stay ellipsoids throughout the deformation history, which axisymmetric cell analysis (Gărăjeu et al., 2000) has shown to be an excellent approximation for axisymmetric tensile load cases. For compressive loading, however, initially spherical pores may evolve into markedly different shapes and contact between the walls of pores tends to play an increasing role as the voids as their volume fraction is more and more reduced (Segurado et al., 2002b). Void size effects (Tvergaard, 1996) and void coalescence (Faleskog and Shih, 1997) introduce additional complexity into studies of the ductile damage and failure of metals.
having collapsed to such an extent that many cell walls or struts are in contact and the void volume fraction has decreased dramatically. No comparable behavior is present under tensile or shear loading.

Periodic microfield methods are generally well suited to studying the thermomechanical behavior of cellular materials. The widely used analytical results of Gibson and Ashby (1988) were derived by analytically studying arrangements of beams (for open cell foams) or plates (for closed cell foams). They give the macroscopic moduli and other physical properties of cellular materials as power laws of the type

\[
\frac{E^*}{E^{(m)}} \propto \left( \frac{\rho^*}{\rho^{(m)}} \right)^n
\]  

in terms of the relative density\(^{104}\), \(\rho^*/\rho^{(m)}\).

In Finite Element based models of cellular materials the solid phase may be described either by continuum elements or by structural elements (shells for the cell walls of closed cell foams and beams for the struts in open cell foams)\(^{105}\). The influence of the gas filling closed cells can also be accounted for within such an FE setting (Mills et al., 2009). In order to model buckling and compaction phenomena, unit cells for cellular materials require explicit provision for handling large deformations of and contact (including self-contact) between cell walls or struts. Boundary conditions must be applied to the unit cells in such a way that they do not interfere with relevant buckling modes. Specifically, models must be sufficiently large so that nontrivial deformation and buckling patterns can develop, or Bloch wave theory (Gong et al., 2005) must be used to account for long wavelength buckling modes\(^{106}\).

The geometrically simplest cellular materials are regular honeycombs, the in-plane behavior of which can be modeled with planar hexagonal cell models. Somewhat less ordered two-dimensional arrangements have been used for studying the crushing behavior of soft woods (Holmberg et al., 1999), and highly irregular planar arrangements, of the type shown in fig. 5.22, may be used to generically investigate aspects of the geometry dependence of the mechanical response of cellular materials.

The modeling of typical closed cell foams, however, requires three-dimensional microgeometries. In the simplest cases generic phase arrangements based of cubic cells, see, e.g.,

\(^{104}\)The exponent \(n\) in eqn. (5.20) allows inferring the dominant local deformation mode of a cellular microgeometry, with \(n=2\) implying deformation by bending and \(n=1\) by axial elongation/compression of struts in open cell foams. The Gibson–Ashby cell for open cell foams gives \(n=2\) and, as a consequence, tends to underestimate the stiffness of actual open-cell foams.

\(^{105}\)At high levels of porosity it is typically necessary to account for the overlap of shell elements at edges and of beam elements at vertices when evaluating the phase volume fractions of the discretized unit cells. When structural finite elements are used in unit cell studies periodicity conditions must obviously be enforced in terms of both translational and rotational degrees of freedom.

\(^{106}\)For perfectly regular structures such as hexagonal honeycombs the minimum size of a unit cell for capturing bifurcation effects can be estimated from extended homogenization theory (Saiki et al., 2002). When symmetry boundary conditions are employed care must be taken that cell walls that may buckle do not coincide with symmetry planes at the boundaries, and periodic contact may have to be provided for if periodicity boundary conditions are used.

121
Figure 5.22: Planar periodic unit cell for studying irregular cellular materials (Daxner, 2003).

Hollister et al. (1991), truncated cubes plus small cubes (Santosa and Wierzbicki, 1998), rhombic dodecahedra, regular tetrakaidecahedra (Grenestedt, 1998; Simone and Gibson, 1998), compare fig. 5.23, as well as Kelvin, Williams and Weaire–Phelan (Kraynik and Reinelt, 1996; Daxner et al., 2007) geometries\(^{107}\) may be used. Analogous regular microgeometries have formed the basis for analytical and numerical “lattice models” of open cell foams, see, e.g., Zhu et al. (1997), Shulmeister et al. (1998) and Vajjhala et al. (2000) as well as fig. 5.24, the struts connecting the nodes of the cellular structures being modeled by beam or solid elements\(^{108}\). In addition, tetrahedral arrangements (Sihn and Roy, 2005) and cubic arrangements of struts with additional “reinforcements” as well as perturbed strut configurations (Luxner et al., 2005) have been covered by such studies. Voronoi tessellations have become a common tool in modeling irregular open and closed cell foams, and random Laguerre tessellations have been proposed for generating periodic multi-cell models (Redenbach, 2009).

The effects of details of the microgeometries of cellular materials (e.g., thickness distributions and geometrical imperfections or flaws of cell walls or struts or the Plateau borders formed at the intersections of cell walls), which can considerably influence the overall behavior, have been a fruitful field of research employing periodic homogenization, see, e.g., Grenestedt (1998) and Daxner (2003). Typically, unit cell analysis of cellular materials with realistic microgeometries is rather complex and numerically demanding due to these

\(^{107}\) In Kelvin, Williams and Weaire–Phelan (Weaire and Phelan, 1994) foams some of the cell walls are spatially distorted in order to minimize the total surface, whereas in polyhedral foams all cell walls are planar. At low solid volume fractions these small distortions lead to noticeable differences in the overall elastic response.

\(^{108}\) Models of foams employing regular arrangements of polyhedral cells are not macroscopically isotropic, compare, e.g., Luxner et al. (2005). A specific issue of beam lattice models for open cell foams are the limitations of Timoshenko beams that typically fully account for nonlinear material behavior in the axial and bending stiffnesses only, but not in shear stiffness, see, e.g., Pettermann and Hüsing (2012).
materials’ tendency to deform by local mechanisms and instabilities. Nevertheless, analytical solutions have been reported for the linear elastic behavior of some simple periodic phase arrangements, see, e.g., Warren and Kraynik (1991) or Sullivan and Ghosn (2009).

The high flexibility of FE-based periodic homogenization has allowed, on the hand, modeling a wide range of material behaviors of cellular materials, among them plasticity, viscoelasticity (Pettermann and Hüsing, 2012) and metal creep (Oppenheimer and Dunand, 2007), and, on the other hand, studying fairly complex microgeometries. An example of the latter describes hollow strut foams produced by coating a precursor cellular material and then removing this “template”. Figure 5.24 shows a model for such a foam that is based on a Weaire–Phelan geometry and was meshed by a combination of solid and shell elements (Daxner et al., 2007).

Figure 5.23: Idealized closed-cell foam microstructure modeled by regular tetrakaidecahedra (truncated octahedra).

Figure 5.24: Weaire–Phelan model of an open cell foam with hollow struts (left) and detail of the FE-model (right), from Daxner et al. (2007).
Due to the inherent X-ray absorption contrast between matrix and voids, cellular materials are well suited to high-resolution computed tomography, giving access to “real structure” microgeometries that can be directly converted into voxel models (compare section 4.3) of open-cell or closed-cell foams\textsuperscript{109}, see, e.g., Maire et al. (2000) or Roberts and Garboczi (2001). Alternatively, image processing methods may be used to generate a surface model of the solid phase in the volume element to allow meshing with “standard” FE techniques (Youssef et al., 2005; Young et al., 2008). Because the microgeometries extracted from the experiments are, in general, not periodic, windowing approaches using homogeneous or mixed uniform boundary conditions, compare chapter 6, tend to be the method of choice for such models, however.

A specific group of cellular materials amenable to PMA modeling are syntactic foams (i.e., hollow spheres embedded in a solid matrix) and hollow sphere foams (in which the spaces between the spheres are “empty”, giving them both an open-cell and a closed cell flavor). Syntactic foams have been studied via axisymmetric or three-dimensional cell models that are based on closest packings of spheres (Rammerstorfer and Böh, 2000; Sanders and Gibson, 2003).

Recent advances in additive manufacturing have led to major interest in architected, lattice or shell micro- and nanostructures, which are periodic by design at the microscale. They are, accordingly, amenable to study by periodic homogenization, compare, e.g., Abueidda et al. (2017). However, for finite samples of finite size boundary effects may play a major role in their behavior, which requires modeling by full structural models (Luxner et al., 2005) or by second order homogenization (Desmoulins and Kochmann, 2017).

A further type of cellular material, trabecular (cancellous, spongy) bone, has attracted considerable modeling interest for more than thirty years\textsuperscript{110}. Cancellous bone shows a wide range of microgeometries, which can be idealized as beam or beam–plate configurations (Gibson, 1985). In studying the mechanical behavior of trabecular bone, large three-dimensional unit cell models based on tomographic scans of actual samples and using voxel-based or smooth discretization schemes have become fairly widely used, see, e.g., Hollister et al. (1991), Hollister et al. (1994) or van Rietbergen et al. (1999).

In addition to periodic homogenization, windowing methods, compare chapter 6, and embedding models, see chapter 7, have been successfully employed for studying cellular materials.

\textsuperscript{109} Roberts and Garboczi (2001) estimated the “systematic discretization error” to be of the order of 10\% in terms of the overall moduli for voxel-based models of elastic open-cell and closed-cell foams.

\textsuperscript{110}Like many materials of biological origin, bone is an inhomogeneous material at a number of length scales. The solid phase of trabecular bone is inhomogeneous and can itself be studied with micromechanical methods.
5.8 Periodic Models for Some Other Inhomogeneous Materials

At the continuum level, the thermomechanical behavior of essentially any inhomogeneous material can be studied by periodic homogenization techniques. For example, the elastic, elastoplastic, creep and damage behaviors of polycrystals, of high speed steels and dual phase steels, of intermetallics, of superalloys, and of graded materials have been the targets of unit cell models. In addition, “smart materials”, such as piezoelectric composites and shape memory alloys, and solid state phase transitions in general have been studied. A comprehensive discussion of the models involved would by far exceed the present scope, so that the present section is limited to a small number of examples.

Generic PMA Models for Clustered, Graded, and Interpenetrating Microgeometries

Many commercially important steels may be viewed as matrix–inclusion-type composites. For example, high speed steels (HSS) contain carbidic inclusions in a steel matrix, with arrangements ranging from statistically homogeneous to highly clustered or layered (meso)geometries. Such materials can be modeled like particle reinforced MMCs. A straightforward and flexible strategy for setting up generic planar model geometries for such studies consists of tiling the computational plane with regular hexagonal cells which are assigned to one of the constituents by statistical or deterministic rules. If required, the shapes of the cells can be modified or randomly distorted and their sizes can be changed to adjust phase volume fractions. Such a Hexagonal Cell Tiling (HCT) concept can be used, e.g., to generate unit cells for analyzing layer structured HSSs, compare (Plankensteiner et al., 1997) and fig. 5.25, and clustered or random arrangements of inhomogeneities in a matrix. HCT and related models of matrix–inclusion topologies are, however, restricted to low reinforcement volume fractions.

Figure 5.25: HCT cell models for a layer structured high-speed steel (left) and a functionally graded material (right).
Related concepts can be used to obtain generic geometries for planar models for studying functionally graded materials\(^{111}\) (FGMs), in which the phase volume fractions run through the full physically possible range (Weissenbek et al., 1997; Reiter et al., 1997), so that there are regions that show matrix–inclusion microtopologies and others that do not, compare fig. 5.25 (right). A further application of such models have been investigations of the microstructure–property relationships in duplex steels, for which matrix–inclusion and interwoven phase arrangements were compared\(^{112}\), see Siegmund et al. (1995) as well as Silberschmidt and Werner (2001). An FFT-based modeling scheme employing HCT-type geometries was developed by Michel et al. (1999) explicitly for studying interpenetrating microgeometries. There is no direct three-dimensional equivalent to HCT models, but unit cells partitioned into cube-shaped or tetrakaidecahedral subregions can be used in an analogous manner to obtain generic microgeometries (which, however, may be numerically expensive if meshes are chosen such that details of the microfields can be resolved). A distantly related, but geometrically much more flexible approach based on tetrahedral regions was reported by Galli et al. (2008).

For topological reasons most interpenetrating phase composites (IPC) must be studied with three-dimensional models. Beyond this restriction, however, fairly standard periodic homogenization approaches can be used for investigating their effective thermomechanical behavior, see, e.g., Dalaq et al. (2016) or Soyarslan et al. (2018).

**PMA Models for Polycrystals**

A large number of materials that are of technological interest are polycrystals and, accordingly, are inhomogeneous at some sufficiently small length scale. In the special case of metals and metallic alloys phenomena associated with plastic flow and ductile failure can be investigated at a number of length scales (McDowell, 2000), some of which are amenable to study by continuum micromechanical methods.

Micromechanical methods, especially analytical self-consistent approaches, have been applied to studying the mechanical behavior of polycrystals since the 1950s. The more recent concept of studying polycrystals by full field approaches, among them periodic homogenization, is quite straightforward in principle — a planar or three-dimensional unit cell is partitioned into suitable subregions that correspond to individual grains\(^{113}\). For these grains appropriate material models must be prescribed, and suitable material as well as orientation parameters must be assigned to them. The generation of appropriate grain geometries may be based, e.g., on Voronoi tessellations using Poisson or hardcore distributed

---

\(^{111}\) In contrast to most other materials discussed in the present report, FGMs are statistically inhomogeneous materials, so that phase arrangements must be non-periodic in at least one direction, for which free-surface boundary conditions are used on the unit cells. Accordingly, volume averaging is of limited value in modeling such materials.

\(^{112}\) In HSSs and FGMs particulate inhomogeneities are present at least in part of the volume fraction range, so that planar analysis must be viewed as a compromise between modeling accuracy and computational requirements. Duplex steels, in contrast, typically show elongated aligned grains, which can be described well by generalized plane strain analysis in the transverse plane.

\(^{113}\) Usually symmetry boundaries passing through grains must be avoided in models of this type because they give rise to “twin-like” pairs of grains that are unphysical in most situations.
seed points, or on modified Voronoi tessellations. Such microgeometries may be linked to uniform and isotropic grain growth that stops where neighboring grains contact each other, may be efficiently generated as well as meshed by appropriate software (Fritzen et al., 2009), and can provide periodic or non-periodic volume elements. Alternatively, the grain geometries can be based on experimental data obtained, e.g., by microscopy and serial sectioning.

Anisotropic elasticity and crystal plasticity models are best suited to describing the material behavior of the individual grains, the orientation of which can be described by stochastic models. The number of grains required for achieving a given accuracy in terms of the macroscopic elastic tensor has been shown to depend on the level of anisotropy of the individual grains (Nygård, 2003). Analysis involving crystal plasticity tends to pose considerable demands on computational resources for three-dimensional models, compare Quilici and Cailletaud (1999). For a general discussion of the issues involved in micromechanical models in crystal plasticity and related fields see, e.g., Dawson (2000) and Roters et al. (2010).

**PMA Models for Two-Phase Single Crystal Superalloys**

Nickel-base single crystal superalloys, which consist of a $\gamma$ matrix phase containing aligned cuboidal $\gamma'$ precipitates, are of considerable importance due to their creep resistance at high temperatures, their main field of application being the hot sections of gas turbines. Because these materials show relatively regular microgeometries that change under load, due to a process known as rafting, there has been considerable interest in micromechanical modeling for elucidating their thermomechanical behavior and for better understanding their microstructural evolution. The elastic behavior of two-phase superalloys can be handled relatively easily by hexahedral unit cells with appropriately oriented anisotropic phases. In the inelastic range, however, the highly constrained plastic flow in the $\gamma$ channels is difficult to describe even by crystal plasticity models, see, e.g., Nouailhas and Cailletaud (1996).

**5.9 Periodic Models Models for Diffusion-Type Problems**

Periodic microfield methods analogous to those discussed in sections 5.1 to 5.8 can be used for studying linear diffusion-type problems of the types mentioned in section 2.8. The Laplace solvers required for numerically-based homogenization in diffusion problems are widely available in FE packages, usually for modeling heat conduction. Specialized solvers are, however, required in some cases, e.g., when studying the frequency dependent dielectric properties of composites via complex potentials, see, e.g., Krakovsky and Myroshnychenko (2002).

For the most common application, thermal conduction, the equivalent to eqn. (5.2) takes the form

$$T(z + c_k) = T(z) + \langle d \rangle c_k,$$

(5.21)

114 Although Voronoi tessellations are commonly used for generating microgeometries for modeling poly-crystals, the results may differ noticeably from actual microgeometries, see, e.g., Lazar et al. (2012).
where the nomenclature of table 2.1 is followed and \( \langle d \rangle \) is the volume averaged temperature gradient, compare also section 2.8. Periodicity boundary conditions can then be expressed in terms of the nodal temperatures as

\[
\Delta T_k = T_{k+} - T_{k-} = T(s_k + c_k) - T(s_k) = \langle d \rangle c_k
\]

(5.22)

in direct analogy to eqn. (5.3). Asymptotic homogenization for thermal conduction was discussed, e.g., by Auriault (1983), Matt and Cruz (2002), or Tang and Yu (2007), and the method of macroscopic degrees of freedom, compare section 5.3, can be formulated in terms of temperatures, thermal gradients and fluxes rather than displacements, strains and stresses (Nogales, 2008). Furthermore, symmetry planes of the phase arrangement that are oriented parallel or normally to far field gradients can be used to specify “symmetry-like” boundary conditions by not constraining temperatures at all or by setting them to a fixed value, respectively. Volume and phase averages of fluxes are best evaluated according to eqn. (4.8). Finite interfacial conductances can be handled by using appropriate interface elements, compare, e.g., Matt and Cruz (2008) or Nogales and Böhm (2008).

There are, however, some intrinsic conceptual difficulties in applying periodic homogenization to transport problems where nonlinear conduction or diffusion behavior of the constituents is involved. Whereas in solid mechanics material nonlinearities are typically formulated in terms of the microscopic stresses and strains, which may be viewed as “generalized gradient” and “generalized flux” fields (compare table 2.1), respectively, nonlinear conductivities and diffusivities in transport problems typically depend on the “direct variable” (or “potential”), e.g., the temperature in heat conduction. As is evident from fig. 5.1, in PMAs the generalized gradients and fluxes are periodic (and have constant averages), whereas the direct variables consist of fluctuating and linearly varying contributions that accumulate from cell to cell. As a consequence, whereas in solid mechanics periodic homogenization involving nonlinear material behavior uses identical material properties for all unit cells, this is in general not the case for transport problems. Accordingly, even though solutions in terms of both microscopic (gradient and flux) fields and homogenized macroscopic conductivities can be obtained from unit cell analysis involving material nonlinearities of the above type, no consistent periodic model material is obtained. Temizer and Wriggers (2011) carried out a detailed analysis of two-scale homogenization for conduction problems and showed that there is a general thermodynamical inconsistency for finite deviations of microscopic temperatures from the macroscopic ones. Accordingly, the use of periodic homogenization for studying nonlinear transport problems in inhomogeneous media has to be viewed with some reservation and windowing or embedding methods, see chapters 6 and 7, may be preferable for such work.

\[\text{As discussed by Duschlbauer et al. (2003a), symmetry planes oriented parallel to the far field gradient act as insulating surfaces, whereas symmetry planes oriented normally to the applied gradient are isothermal planes. Gradient fields are symmetric with respect to the latter planes, temperature fields are antysymmetric and flux fields non-symmetric.}\]

\[\text{The fact that periodicity implies that the direct variable can take values from } -\infty \text{ to } +\infty \text{ leads to another inconsistency in PMAs for thermal conduction, where the variable } T \text{ has a thermodynamical lower bound at absolute zero. However, this issue does not appear to have any practical repercussions.}\]
Chapter 6

Windowing Approaches

The aim of windowing methods lies in obtaining estimates for or bounds on the macroscopic properties of inhomogeneous materials on the basis of — typically non-periodic — volume elements that are referred to as mesoscopic test windows, observation windows or, in short, “windows”. These volume elements typically are too small to be RVEs, so that the results of windowing analysis tend to pertain to specific samples rather than to a material and are, accordingly, referred to as apparent (rather than effective) material properties.

Windowing implicitly assumes that the “parent material” is statistically homogeneous, so that windows can be extracted from it at random positions. For convenience, windows are often chosen to be rectangles or right hexahedra, but other shapes may be used just as well. If the parent material is known (or at least assumed) to be macroscopically isotropic, windows are best extracted at random orientations, as shown in fig. 6.1. For anisotropic materials, however, the orientations of windows must either be taken into consideration in taking the samples or it must be accounted for explicitly in processing the results of the analysis.

Figure 6.1: Schematic depiction of a composite that is statistically transversally isotropic and of four rectangular windows of equal size in the transverse plane.
Windowing methods subject window-type volume elements to statically uniform boundary conditions (SUBC), eqn. (4.3), kinematically uniform boundary conditions (KUBC), eqn. (4.4), or mixed uniform boundary conditions (MUBC), eqn. (4.5), for obtaining lower estimates, upper estimates and estimates lying between the above, respectively.

**Macrohomogeneous Boundary Conditions**

SUBC and KUBC are collectively referred to as macrohomogeneous boundary conditions. Compliance tensors obtained with SUBC and elasticity tensors obtained with KUBC provide lower and upper estimates on the macroscopic elastic tensors of volume elements, respectively (Nemat-Nasser and Hori, 1993). For volume elements of equal size, ensemble averages of these lower and upper estimates give rise to lower and upper bounds on the overall apparent tensors (Huet, 1990); for unequally sized windows weighted averages must be used. These bounds are sometimes referred to as mesoscale bounds and provide information on the macroscopic properties pertinent to a collection of windows of given size. For further discussions of these issues see, e.g., Ostoja-Starzewski (2008).

By definition, if the windows are of sufficient size to be proper RVEs the lower and upper estimates and bounds on the overall elastic properties must coincide (Huet, 1990; Sab, 1992), defining the effective behavior. Accordingly, hierarchies of bounds that are generated from sequences of sets of sub-RVE testing volume elements of different sizes can be used for assessing the size of proper representative volume elements. Actually achieving the coincidence of lower and upper mesoscale bounds has been found very difficult in practice, however. This is at least partly due to boundary layers, see, e.g., Pahr and Böhmer (2008), that form due to the interaction of the uniform B.C. with phase boundaries intersecting the volume elements’ surfaces, compare the remarks in section 4.2. Consequently, using volumes that are shaped such that their boundaries stay within the contiguous phase of matrix–inclusion composites, which reduces boundary perturbations, has been found to improve the convergence of hierarchies of mesoscale bounds to a considerable extent (Salmi et al., 2012). This expedient, however, may bias phase volume fractions in the windows and may give rise to difficulties in identifying proper RVEs.

The concept of generating lower and upper estimates by windowing using SUBC and KUBC can be shown to be also valid in the context of nonlinear elasticity and deformation plasticity (Jiang et al., 2001). Discussions of windowing bounds in finite strain elasticity, in viscoelasticity and in incremental plasticity settings were given by Khisaeva and Ostoja-Starzewski (2006), Zhang and Ostoja-Starzewski (2015) as well as Li and Ostoja-Starzewski (2006), respectively. Windowing methods are also directly applicable to conduction problems, with macrohomogeneous flux and gradient boundary conditions giving rise to lower and upper estimates, respectively, as well as mesoscale bounds for the conductivity tensor.

Generally, if the individual windows are small, considerable variations in the phase volume fractions will typically be present in different realizations and the macroscopic material symmetry must be expected to be subject to considerable perturbations. Furthermore, for

---

117 Hierarchies of bounds for a given microgeometry can be generated by using series of windows of increasing size, which allows to assess the dependence of the predicted overall moduli on the size of the windows.
relatively small windows, within which the phase arrangement deviates significantly from statistical homogeneity, different boundary conditions, especially SUBC and KUBC, tend to give rise to marked differences in the distributions of the microfields, especially the plastic strains in elastoplastic models. When larger windows are used, however, increasingly similar statistical distributions of the microstresses and microstrains are obtained (Shen and Brinson, 2006).

For most materials macrohomogeneous boundary conditions are fairly easy to implement in a Finite Element setting. However, in general homogeneous macrostrains cannot be prescribed along boundaries that intersect rigid inhomogeneities and homogeneous tangential tractions cannot be imposed along boundaries that intersect voids.

**Mixed Uniform Boundary Conditions**

Equations (4.5) are fulfilled by a range of different sets of MUBC, resulting in different estimates for the apparent macroscopic elastic tensors, all of which lie between the lower and upper estimates provided by the macrohomogeneous boundary conditions (Hazanov and Huet, 1994). A specific set of mixed uniform boundary conditions for volume elements that avoids prescribing non-zero boundary tractions was proposed by Pahr and Zysset (2008) for obtaining accurate apparent elastic tensors of volume elements of cellular materials. Table 6.1 lists these conditions which corresponding to six linearly independent mechanical load cases for use with three-dimensional volume elements that have the shape of right hexahedra aligned with the coordinate system. The distances \( l_1, l_2 \) and \( l_3 \) correspond to cell’s edge lengths in the 1-, 2- and 3-directions, compare Fig. 5.4. The components of the prescribed strain tensor are denoted as \( \varepsilon_{ij}^a \) and those of the prescribed traction vector as \( \tau_i^a \).

---

**Table 6.1:** The six linearly independent uniform strain load cases making up the periodicity compatible mixed uniform boundary conditions (PMUBC) proposed by Pahr and Zysset (2008) and the pertinent boundary conditions for loading by a uniform temperature; East, West, North, South, Top, and Bottom denote the faces of a right hexahedral volume element, compare fig. 5.4, and the the distances \( l_i \) correspond to the cell’s edge lengths in the 1-, 2- and 3-directions.

<table>
<thead>
<tr>
<th>Case</th>
<th>East</th>
<th>West</th>
<th>North</th>
<th>South</th>
<th>Top</th>
<th>Bottom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile 1</td>
<td>( u_1 = \varepsilon_{11}^a l_1 / 2 )</td>
<td>( u_2 = 0 )</td>
<td>( u_3 = 0 )</td>
<td>( u_4 = 0 )</td>
<td>( u_5 = 0 )</td>
<td>( u_6 = 0 )</td>
</tr>
<tr>
<td>Tensile 2</td>
<td>( \tau_1^a = \tau_2^a = 0 )</td>
<td>( \tau_3^a = \tau_4^a = 0 )</td>
<td>( \tau_5^a = \tau_6^a = 0 )</td>
<td>( \tau_1^a = \tau_2^a = 0 )</td>
<td>( \tau_3^a = \tau_4^a = 0 )</td>
<td>( \tau_5^a = \tau_6^a = 0 )</td>
</tr>
<tr>
<td>Tensile 3</td>
<td>( u_1 = 0 )</td>
<td>( u_2 = 0 )</td>
<td>( u_3 = 0 )</td>
<td>( u_4 = 0 )</td>
<td>( u_5 = \varepsilon_{13}^a l_1 / 2 )</td>
<td>( u_6 = -\varepsilon_{23}^a l_1 / 2 )</td>
</tr>
<tr>
<td>Shear 12</td>
<td>( u_2 = \varepsilon_{23}^a l_1 / 2 )</td>
<td>( u_3 = -\varepsilon_{31}^a l_1 / 2 )</td>
<td>( u_4 = 0 )</td>
<td>( u_5 = 0 )</td>
<td>( u_6 = 0 )</td>
<td>( u_7 = 0 )</td>
</tr>
<tr>
<td>Shear 13</td>
<td>( u_3 = \varepsilon_{31}^a l_1 / 2 )</td>
<td>( u_4 = 0 )</td>
<td>( u_5 = 0 )</td>
<td>( u_6 = 0 )</td>
<td>( u_7 = 0 )</td>
<td>( u_8 = 0 )</td>
</tr>
<tr>
<td>Shear 23</td>
<td>( u_4 = 0 )</td>
<td>( u_5 = 0 )</td>
<td>( u_6 = 0 )</td>
<td>( u_7 = 0 )</td>
<td>( u_8 = 0 )</td>
<td>( u_9 = 0 )</td>
</tr>
<tr>
<td>Thermal</td>
<td>( u_1 = 0 )</td>
<td>( u_2 = 0 )</td>
<td>( u_3 = 0 )</td>
<td>( u_4 = 0 )</td>
<td>( u_5 = 0 )</td>
<td>( u_6 = 0 )</td>
</tr>
<tr>
<td>Expansion</td>
<td>( \tau_1^a = \tau_2^a = 0 )</td>
<td>( \tau_3^a = \tau_4^a = 0 )</td>
<td>( \tau_5^a = \tau_6^a = 0 )</td>
<td>( \tau_7^a = \tau_8^a = 0 )</td>
<td>( \tau_9^a = \tau_10^a = 0 )</td>
<td>( \tau_11^a = \tau_12^a = 0 )</td>
</tr>
</tbody>
</table>
Compared to macrohomogeneous boundary conditions the PMUBC lead to much faster convergence of the predicted homogenized properties in terms of the size of the volume elements, similarly to periodicity boundary conditions, compare section 5.2, and to certain embedding schemes, compare chapter 7.

The use of these MUBC is not limited to cellular materials, and when applied to periodic volume elements of orthotropic effective behavior, they were found to give very similar predictions for the macroscopic elasticity tensor as does periodic homogenization (Pahr and Böhm, 2008). This led to their being named periodicity compatible mixed uniform boundary conditions (PMUBC). Their behavior with periodic microgeometries indicates that PMUBC can also be used to advantage for obtaining estimates from non-periodic volume elements, at least when the sub-orthotropic contributions to the overall symmetry are relatively small. Accordingly, these boundary conditions offer an attractive option for evaluating estimates of the macroscopic elastic behavior on the basis of TVEs.

The concept of periodicity compatible mixed uniform boundary conditions can be extended to thermoelasticity by adding a load case that constrains all displacements normal to the faces of the volume element, sets all in-plane tractions to zero, and applies a uniform temperature increment \( \Delta T \), see table 6.1. This allows to evaluate the volume averaged specific thermal stress coefficient tensor \( \langle \lambda \rangle \), from which the apparent thermal expansion tensor can be obtained as \( \alpha = -C\langle \lambda \rangle \), the apparent compliance tensor \( C \) being evaluated from the first six equations of table 6.1.

Mixed uniform boundary conditions for diffusion-like problems were discussed, e.g., by Jiang et al. (2001). Boundary conditions that show an analogous behavior to the PMUBC in elasticity were reported by Jiang et al. (2002a) for thermal conduction in two-dimensional orthotropic periodic media, see table 6.2. The handling of finite interfacial conductances in the context of windowing was studied by Nogales (2008).

Table 6.2: The three linearly independent uniform gradient load cases making up the periodicity compatible mixed uniform boundary conditions (PMUBC) in thermal conduction; East, West, North, South, Top, and Bottom denote the faces of a hexahedral volume element, compare fig. 5.4, the \( d_i^a \) are the components of the applied temperature gradient, and the the distances \( l_i \) can be obtained by scaling the cell’s edge lengths, \( c_i \), by a suitable common factor.

<table>
<thead>
<tr>
<th>Thermal 1</th>
<th>East</th>
<th>West</th>
<th>North</th>
<th>South</th>
<th>Top</th>
<th>Bottom</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( T = \frac{d_1^a l_1}{2} )</td>
<td>( T = -\frac{d_1^a l_1}{2} )</td>
<td>( q_2 = 0 )</td>
<td>( q_3 = 0 )</td>
<td>( q_4 = 0 )</td>
<td>( q_5 = 0 )</td>
</tr>
<tr>
<td>Thermal 2</td>
<td>( q_1^1 = 0 )</td>
<td>( q_1^2 = 0 )</td>
<td>( T = \frac{d_2^a l_2}{2} )</td>
<td>( T = \frac{d_2^a l_2}{2} )</td>
<td>( q_4^2 = 0 )</td>
<td>( q_5^2 = 0 )</td>
</tr>
<tr>
<td>Thermal 3</td>
<td>( q_1^3 = 0 )</td>
<td>( q_1^4 = 0 )</td>
<td>( q_2^3 = 0 )</td>
<td>( q_3^3 = 0 )</td>
<td>( T = \frac{d_3^a l_3}{2} )</td>
<td>( T = \frac{d_3^a l_3}{2} )</td>
</tr>
</tbody>
</table>

In contrast to periodicity and macrohomogeneous boundary conditions, the PMUBC listed in table 6.1 are strictly limited to handling these specific load cases if nonlinear behavior is present and the superposition principle no longer holds. Accordingly, general load paths in stress and strain space cannot be followed, so that in nonlinear regimes this type of model is restricted to materials characterization, compare (Pahr and Böhm, 2008).
Certain MUBC, e.g., ones that handle macroscopic uniaxial loading by specifying normal displacements plus zero tangential tractions for one pair of faces and setting all traction components to zero for all other faces, compare, e.g., Galli et al. (2008), can be interpreted physically as describing the behavior of small, inhomogeneous samples\textsuperscript{118} as discussed in section 1.5. Due to the weaker boundary constraints such sample-type MUBC can be expected to predict more compliant macroscopic behavior than do the periodicity boundary conditions discussed in section 5.2 or the PMUBC listed in table 6.1.

Important strengths of windowing methods lie in providing an approach to studying the behavior of non-periodic volume elements and in being considerably easier to handle than PMAs for periodic volume elements. The main reasons for the latter point are that the meshing of homologous faces is not an issue in windowing and multi-point constraint equations are not required.

\textsuperscript{118}In contrast to periodicity BC, these specific MUBC give different results for volume elements made up of a number “base units” compared to models consisting of a single unit. This is in keeping with their interpretation as “structural” rather than “material” models. If the tangential strains rather than the tangential stresses are set to zero for the pair of faces controlling the macroscopic axial deformation, the physical interpretation is that of a sample “sticking” to the base plates, implying the presence of strain gradients at the length scale of the sample. Such models are not capable of describing a proper uniaxial stress load case.
Chapter 7
Embedding Approaches and Related Models

Embedded Cell Approaches (ECAs) aim at predicting the microfields in specific, geometrically highly resolved regions of models of inhomogeneous materials or structures. Such models consist of two regions, as is evident in figs. 7.1 and 7.2. On the one hand, there is a core (or “local heterogeneous region”) consisting of a discrete microstructure (“motif”), which can range from rather simple to highly complex phase arrangements. On the other hand, there is a homogeneous outer region (“embedding region”, “frame”, “effective region”) into which the core is embedded and which serves mainly for transmitting the applied loads. Embedding strategies tend to give rise to relatively complex models, but they avoid the main drawback of PMAs, viz., the requirement that the microgeometry and all microfields must be periodic.

An intrinsic feature of embedding models are boundary layers that occur at the “interfaces” between the core and the embedding region and perturb the local stress and strain fields\(^{119}\), see, e.g., Harper et al. (2012) and compare section 4.2. Provision must be made to keep regions of specific interest, such as crack tips or process zones, at a sufficient distance from the boundary layers\(^{120}\), which, in turn, implies that cores must exceed some minimum size in order to provide useful results.

Most of the embedding approaches reported in the literature for tasks related to continuum micromechanics fall into two groups.

\(^{119}\)The interfaces between core and embedding region are a consequence of the modeling approach and do not have any physical background or significance. In elasticity such boundary layers typically have a thickness of, say, the distance between the centers of neighboring inhomogeneities, but they may be longer ranged for nonlinear material behavior.

\(^{120}\)Boundary layers and embedding regions may interfere with extended regions of concentrated strains, with shear bands, or with the growth of damaged regions. Such difficulties can only be avoided or mitigated by choosing a sufficiently large core region. In dynamic models the boundaries between core and embedding region may lead lead to the reflection and/or refraction of waves. Special transition layers can, however, be introduced to mitigate boundary layer effects in the latter case.
Embedding Region with Self-Consistently Determined Response

One group of embedding schemes employ the homogenized thermomechanical response of the core for determining the effective behavior of the surrounding medium via self-consistent procedures. By applying suitable uniform far field loads to the outer boundaries of such an embedding layer, as sketched in Fig. 7.1, models of this type can be used for carrying out scale transitions in terms of homogenization and, with some precautions, localization.

![Figure 7.1: Schematic depiction of the arrangement of core and embedding region in a self-consistent embedded cell model subjected to a uniform far-field load.](image)

As mentioned in section 4.2 self-consistent embedding schemes employing macrohomogeneous or periodic boundary conditions\(^\text{121}\) provide predictions for the effective elasticity tensors that lie between the lower and upper estimates on the apparent tensors and, by extension, within the mesoscale bounds discussed in chapter 6. Such approaches, in which the embedding layer mainly serves for smoothening out fluctuations at the core’s boundaries have been termed the “Window Method” by some authors (Krabbenhøft et al., 2008; Temizer et al., 2013), and the designator “augmentation layer” was used for the embedding region in an earlier, related approach (Babuška et al., 1999). In a similar vein, the use of a self-consistent embedding “buffer layer” has been proposed for models in which periodicity boundary conditions are applied to non-periodic volume elements (Makowski et al., 2013).

Because self-consistent iterations are required, for a given inhomogeneous volume element self-consistent embedding methods tend to be more expensive numerically than are periodic homogenization and windowing analysis (in the sense of chapter 6). For elastic analysis, however, the number of required iterations tends to be fairly low and the approach

\(^{121}\)In such schemes the full core (including the boundary layers) must be used for evaluating the effective responses (Temizer et al., 2013).
has been shown to be unconditionally stable by Salit and Gross (2014). Some issues in choosing the width of the embedding layer for linear analysis were discussed by Temizer et al. (2013).

The use of self-consistency procedures is, of course, predicated on the availability of suitable parameterizable constitutive laws for the embedding material that can follow the core’s (instantaneous) homogenized behavior with high accuracy for all load cases and loading histories considered. This requirement can typically be fulfilled easily in the linear range, see e.g., Yang et al. (1994) and Chen (1997), but may lead to considerable complexity when at least one of the constituents shows elastoplastic or viscoplastic material behavior. Accordingly, approximations may have to be used (the consequences of which may be difficult to assess in view of the nonlinearity and path dependence of the above material behaviors) and models of this type are best termed “quasi-self-consistent schemes”. Such approaches were discussed, e.g., by Bornert et al. (1994) and by Dong and Schmauder (1996).

Self-consistent and quasi-self-consistent embedding models are not suitable for handling “strong features”, such as localized cracks of the type shown in fig. 7.2, within the core and their physical interpretation may be difficult when the core includes subregions subject to damage.

**Embedding Region with Prescribed Response**

In the other group of embedding methods the behavior of the outer region is described via appropriate, pre-defined, “smeared-out” constitutive models. These typically take the form of semi-empirical or micromechanically based constitutive laws that are prescribed a priori for the embedding zone and are chosen to represent the appropriate (usually damage-free) material behavior. This way, conceptually simple models are obtained that are very well suited to studying local phenomena such as the stress and strain distributions in the vicinity of crack tips (Aoki et al., 1996), around local defects (Xia et al., 2001) or at macroscopic interfaces (Ayyar et al., 2008). The constitutive model and the pertinent material parameters required for the embedding layer may be obtained, e.g., from micromechanical modeling or from experiments (Ayyar et al., 2008).

The description chosen for the embedding region must reflect the material symmetry, e.g., for metal matrices reinforced by continuous fibers the yield surface, the flow rule and the hardening behavior must account for these materials’ marked anisotropy in the elastoplastic regime.

In some cases, e.g., problems involving macroscopic interfaces, core and embedding region may consist of multiple microstructures and materials, respectively. When the core is used for studying damage and failure, its boundary regions should be excluded from such behavior in order to avoid the perturbed stress and strain fields there. Furthermore, it is worth noting that restricting failure or localized damage to the core may restrict the generality of models, compare, e.g., Bao (1992).
interfaces in composites (Chimani et al., 1997), the growth of cracks in inhomogeneous materials (van der Giessen and Tvergaard, 1994; Wulf et al., 1996; Motz et al., 2001; Mishnaevsky, 2004; González and LLorca, 2007) or damage due to the processing of composites (Monaghan and Brazil, 1997). A related approach applies far field strains obtained from an unperturbed periodic model subjected to an appropriate loading history as boundary conditions to a region containing a local microstructural perturbation, compare Aboudi and Ryvkin (2012).

In applications of the type discussed above, loads may be applied or displacement boundary conditions may be specified that impose a far field behavior obtained from a suitable analytical or numerical solution (e.g., the displacements describing the far field of a crack tip in elasticity or small-scale plasticity) pertinent to the problem under study. Alternatively, the embedding region may be chosen to represent a structure or sample to be considered, with the core zooming in on a detail of special interest. Such approaches, which may be viewed as a type of concurrent multiscale modeling scheme, allow complete specimens or components to be studied via “simulated experiments” as sketched in fig. 7.2.

Figure 7.2: Schematic depiction of the arrangement of core and embedding region in an embedded cell model of a tensile test specimen.

Accordingly, whereas embedding models making use of prescribed properties for the embedding region are by design not geared towards homogenization analysis, they may be viewed as methods specialized towards localization. At present, they arguably are the most versatile and powerful micromechanical tool for such tasks124.

124Embedding analysis of this kind can be used without intrinsic restrictions at or in the vicinity of free surfaces and interfaces, it can handle gradients in composition and loads, and it can be employed for studying the interaction of macrocracks with microstructures. Also, the requirement of sufficiently separated length scales, compare section 1.1, does not necessarily apply to embedded models. Care is, however, required, with highly nonlinear analyses, especially with models involving damage, where the inherent perturbations at boundaries between core and embedding region may trigger spurious behavior.
Other Embedding and Related Models

A further approach to embedding uses discrete microstructures in both the core region and in the surrounding material, the latter, however, being discretized by a much coarser FE mesh, see, e.g., Sautter et al. (1993). Such models, which are also referred to as “fine mesh window” approaches (Podgórski, 2011) and are related to descriptions of full samples or structures that contain a refined mesh in some region(s) of interest, can avoid boundary layers to a considerable extent by using appropriately graded meshes. They tend, however, to be rather expensive computationally. Mesh superposition techniques, which use a coarse mesh over the macroscopic model of some structure or sample together with a geometrically independent, much finer mesh in regions of interest (Takano et al., 2001) are conceptually similar to the above modeling strategy. Also, part of the embedding region may be modeled similarly to the core and used as a sacrificial region serving to insulate the core from the boundary layer (Harper et al., 2012).

A closely related modeling strategy makes use of submodeling techniques in which an inhomogeneous micromodel is weakly or strongly coupled to a homogeneous macro model via appropriate coupling conditions (which may conveniently be implemented via the boundary conditions of the micromodel), compare, e.g., Heness et al. (1999) or Váradi et al. (1999). When strongly perturbing microgeometrical features, such as crack tips, are present in the submodel, it is important to make sure the latter is large enough to keep them at a sufficient distance from the boundary layer.

Like the model shown in Fig. 7.2, the approaches discussed above are mainly aimed at localization and they are related to the “inhomogeneous structure models” discussed in section 1.5.

Additional Remarks

When a periodic multi-inhomogeneity volume element is used as the core in a self-consistent or quasi-self-consistent embedding scheme, the concomitant relaxation of the periodicity constraints tends to make the overall responses of the embedded configuration softer compared to periodic homogenization of the periodic arrangement alone (Bruzzi et al., 2001)\textsuperscript{125}.

Some analytical methods such as classical and generalized self-consistent schemes, see section 2.3, can obviously be viewed as embedding schemes that combine relatively simple cores with self-consistently defined material behavior of the embedding region. When geometrically more complex cores are considered in ECA-like frameworks, however, numerical engineering methods are best suited for resolving the discrete phase arrangements in the core region, compare Bornert (1996).

\textsuperscript{125} In the study mentioned above, the difference between PMA and ECA results is relatively small. Identical responses can be expected from the two types of model if the core is described by a proper RVE (so that by definition the boundary conditions do not play a role) and the embedding material can fully describe the homogenized behavior of the core.
The core and embedding regions may be planar, axisymmetric or fully three-dimensional, and symmetries present in the geometries can, as usual, be made use of for reducing the size of the model, compare Xia et al. (2001). Effective and phase averaged stresses and strains from embedded cell models are best evaluated via eqn. (1.4) or its equivalents, and it is typically good practice to use only the central regions of the core for this purpose in order to avoid perturbations from the boundary layers.

All embedding techniques discussed above can also be used in analogy for studying the thermal conduction behavior of inhomogeneous materials.
Chapter 8
Hierarchical and Multi-Scale Models

The micromechanical methods discussed in chapters 2, 5, and 6 are designed for handling a single scale transition between a lower and a higher length scale ("microscale" and "macroscale"), overall responses being obtained by homogenization and local fields by localization. Many inhomogeneous materials, however, show more than two clearly distinct characteristic length scales, typical examples being laminated and woven composites (compare section 5.4), materials in which there are well defined clusters of particles, as well as most biomaterials. In such cases an obvious modeling strategy is a hierarchical approach that uses a sequence of scale transitions, i.e., the material response at any given length scale is described on the basis of the homogenized behavior of the next finer one\(^{126}\). Figure 8.1 schematically shows such a hierarchical model for a particle reinforced composite with a clustered mesostructure.

Figure 8.1: Schematic representation of a hierarchical approach to studying a material consisting of clustered inhomogeneities in a matrix. Two scale transitions, macro ←→ meso and meso ←→ micro, are used.

A hierarchical model can be viewed as involving a sequence of scale transitions, and suitable micromechanical models, i.e., mean-field, unit cell, windowing and, to some extent,

\(^{126}\)Describing the material behavior at lower length scales by a homogenized model implies that characteristic lengths differ by more than, say, an order of magnitude, so that valid homogenization volumes can be defined (there may, however, be exceptions in the case of linear material behavior). Employing hierarchical approaches within “bands” of more or less continuous distributions of length scales, as can be found, e.g., in some metallic foams with highly disperse cell sizes, is problematic.
embedding approaches, may be used as “building blocks” at any level within hierarchical schemes. Such hierarchical modeling strategies have the additional advantage of allowing the behavior of the constituents at all lower length scales to be assessed via the corresponding localization relations. Multiscaling methods can be classified into concurrent and sequential approaches (Chakraborty and Rahman, 2008; Tadmor and Miller, 2011), with the former requiring the physical problem to be studied simultaneously at all length scales involved.

Among the continuum mechanical hierarchical descriptions of the thermomechanical behavior of inhomogeneous materials reported in the literature, some have combined mean-field methods at the higher length scale with mean-field (Hu et al., 1998; Tszeng, 1998) or periodic microfield approaches (González and LLorca, 2000) at the lower length scale. A common strategy for hierarchical modeling, however, uses Finite Element based unit cell or embedding methods at the topmost length scale, which implies that the homogenized material models describing the lower level(s) of the hierarchy must be proper constitutive laws (i.e., material descriptions that are capable of handling any loading conditions and any loading history) that are evaluated at each integration point of the discretized model. For this purpose, either explicit, micromechanically-based constitutive models may be used or micromechanical models may be run at each integration point, the results of which provide “implicit” descriptions of the local material behavior. The models used for such purposes may be analytical, semi-analytical or numerical.

For the elastic range, generating a micromechanically based constitutive description at each integration point of the “macroscopic” model typically does not pose massive computational requirements—once the appropriate homogenized elasticity and thermal concentration tensors have been generated any load case can be handled, often in a sequential mode. For simulating the thermomechanical response of inelastic inhomogeneous materials, however, essentially a full micromechanical submodel has to be maintained and solved for at each integration point, which amounts to a fully concurrent procedure. Although within such a framework the use of sophisticated numerical models at the lower length scales tends to be an expensive proposition in terms of computational requirements, this concept has elicited major research interest, especially within the framework of multiscale models (Belsky et al., 1995; Ghosh et al., 1996; Lee and Ghosh, 1996; Zohdi et al., 1996; Smit et al., 1998; Feyel and Chaboche, 2000; Fish and Shek, 2000; Ibrahimbegović and Marković, 2003; Moës et al., 2003). Pertinent overviews were given by Ghosh et al. (2001), Kanouté et al. (2009), Geers et al. (2010) as well as Nguyen et al. (2011).

It is worth noting that in models of this type (which are often referred to as FE² models) no explicit material law is used on the macroscale, the full constitutive behavior

---

127 Decomposition techniques can be used to formulate the problems at the finer length scales in such a way that they are well suited for parallel processing, allowing the development of computationally highly efficient multi-scale procedures, see, e.g., Oden and Zohdi (1997).

128 Some approaches have been reported that aim at decoupling the scales by parameterizing results from microscopic unit cell analysis and using this data together with an interpolation scheme as an “approximate constitutive model”, see, e.g., Terada and Kikuchi (1996), Gänsier (1998) or Schrefler et al. (1999). The main difficulty in such sequential modeling strategies lies in handling load history and load path dependences to account for elastoplastic constituents and/or microscopic damage to the constituents. Appropriate representation of general multiaxial stress and/or strain states also tends to be a challenge.
being determined concurrently at the microscale. Arguably, such approaches at present represent the most sophisticated application of the concepts of continuum micromechanics.

Lower (but by no means negligible) computational costs can be achieved by using mean-field models in lieu of constitutive models at the integration point level. Applications to elastoplastic composites have included incremental Mori–Tanaka methods, see, e.g., Pettermann (1997) and mean-field based versions of the Transformation Field Analysis, compare, e.g., Fish and Shek (1998) or Plankensteiner (2000). In addition, the semi-analytical nonuniform TFA has been employed for this purpose (Michel and Suquet, 2004). Figure 8.2 shows a result obtained by applying a multiscale approach that uses an incremental Mori–Tanaka method, compare section 2.5, at the integration point level of a two-dimensional meso-scale unit cell model for describing the elastoplastic behavior of a cluster-structured high speed steel. The particle-rich and particle-poor regions used for the description at the mesoscale are treated as particle reinforced MMCs with appropriate reinforcement volume fractions. Such a model not only predicts macroscopic stress–strain curves, but also allows the mesoscopic distributions of phase averaged microscopic variables to be evaluated, compare (Plankensteiner, 2000). For an example of an Extended Finite Element method at the upper length scale coupled to a semi-analytical model at the lower one see Novák et al. (2012).

![Figure 8.2: Phase averaged microscopic equivalent plastic strains in the matrix within the inhomogeneity-poor regions of a cluster-structured high speed steel under mechanical loading as predicted by a mesoscopic unit cell model combined with an incremental Mori–Tanaka model at the microscale (Plankensteiner, 2000).](image)

Alternatively, multiscale approaches may be rather “loosely coupled”, essentially linking together quite different models to obtain an overall result, see, e.g., (Onck and van der Giessen, 1997).

A further development are concurrent multi-scale methods in which the computational domain is adaptively split into regions resolved at the appropriate length scale. In an FE-based hierarchical framework developed by Ghosh et al. (2001) “non-critical” regions are
described by continuum elements with pre-homogenized material properties, and for zones of clearly nonlinear behavior PMA models are automatically activated at the integration points to monitor the phase behavior at the microscale. If local behavior exceeding the capabilities of periodic homogenization is detected (e.g., progressive damage, mesoscopic interfaces, free edges), embedded models of the fully resolved microgeometries may be activated at the appropriate positions, compare Raghavan and Ghosh (2004). Such strategies allow detailed studies of critical regions in inhomogeneous materials, e.g., near free edges. In an alternative approach, homogenization models employing generalized continua have been proposed for handling regions close to free surfaces and zones with high local field gradients (Feyel, 2003).

Finally, it is worth noting that hierarchical and multi-scale approaches are not limited to using the “standard” methods of continuum micromechanics as discussed above. Especially the capability of the Finite Element method of handling highly complex constitutive descriptions has been used to build hierarchical approaches that employ, among others, material models based on crystal plasticity (McHugh et al., 1993) and discrete dislocation plasticity (Cleveringa et al., 1997).

The linking of continuum and atomistic descriptions has been an important goal of modeling work for a considerable time, involving, e.g., concurrent homogenization schemes, compare Curtin and Miller (2003). The equivalent of homogenization in such settings is often referred to as “upwards linking”, that to localization being known as “downward linking”.

143
Chapter 9
Closing Remarks

The research field of continuum micromechanics of materials has enjoyed considerable success in the past four decades in furthering the understanding of the thermomechanical behavior of inhomogeneous materials and in providing predictive tools for engineers and materials scientists. However, its methods are subject to some practical limitations that should be kept in mind when employing them.

All the methods discussed in chapters 2 to 6 implicitly use the assumption that the constituents of the inhomogeneous material to be studied can be treated as homogeneous at the lower length scale, which, of course, does not necessarily hold. When the length scale of the inhomogeneities in a constituent is much smaller than the length scale of the phase arrangement to be studied, the above assumption is valid and hierarchical as well as multiscale models ("successive homogenization") as discussed in chapter 8 can be used (Carrère, 2001). However, no rigorous theory appears to be available at present for handling scale transitions in materials that do not fulfill the above requirement (e.g., particle reinforced composites in which the grain size of the matrix is comparable to or even larger than the size of the particles\textsuperscript{129}). Typically the best that can be done in such cases is either to use sufficiently detailed models with resolved microgeometry (potentially, a computationally very expensive proposition) or to employ homogenized phase properties and be aware of the approximation that is introduced.

A major practical difficulty in the use of continuum micromechanics approaches (which in many cases is closely related to the questions mentioned above) has been identifying appropriate constitutive models and obtaining the required material parameters for the constituents. Typically, available data pertain to the behavior of the bulk materials (as measured from homogeneous macroscopic samples), whereas the actual requirement is for parameters and, in some cases, constitutive theories that describe the in-situ response of the constituents at the microscale. For example, with respect to MMCs, on the one hand, there is considerable evidence that classical continuum plasticity theories (in which there is no absolute length scale) cannot adequately describe the behavior of metallic materials in the presence of strong strain gradients, where geometrically necessary dislocations can markedly influence the material behavior (Hutchinson, 2000). On the other hand,

\textsuperscript{129}For this special case unit cell or embedding models employing anisotropic or crystal plasticity models for the phases may be used which, however, can become very large when the statistics of grain orientation are accounted for.
it is well known that the presence of reinforcements can lead to accelerated aging and refinement of the grain size in the matrix (“secondary strengthening”). Furthermore, in many cases reasonably accurate material parameters are essentially not available (e.g., strength parameters for most interfaces in inhomogeneous materials). In fact, the dearth of dependable material parameters is one of the reasons why predictions of the strength of inhomogeneous materials by micromechanical methods tend to be a considerable challenge.

Another point that should be kept in mind is that continuum micromechanical descriptions in most cases do not have absolute length scales unless a length scale is introduced explicitly, typically via the constitutive model(s) of one (or more) of the constituents\(^{130}\). In this vein, absolute length scales, on the one hand, may be provided explicitly via discrete dislocation models (Cleveringa et al., 1997), via gradient or nonlocal constitutive laws, see, e.g., Tomita et al. (2000) or Niordson and Tvergaard (2001), or via (nonlocal) damage models. On the other hand, they may be introduced in an ad-hoc fashion, e.g., by adjusting the phase material parameters to account for grain sizes via the Hall–Petch effect. Analysts should also be aware that absolute length scales may be introduced inadvertently into a model by mesh dependence effects of discretizing numerical methods, a “classical” example being strain localization due to softening material behavior of a constituent. When hierarchical or multi-scale models are used special care may be necessary to avoid introducing inconsistent length scales at different modeling levels.

In addition it is worth noting that usually the macroscopic responses of inhomogeneous materials (and, as a consequence of eqns. (2.8) to (2.14), the phase averages of the microfields) are much less sensitive to the phase arrangement (and to modeling approximations) than are the distributions of the microfields. Accordingly, whereas good agreement in the overall behavior of a given model with “benchmark” theoretical results or experimental data typically indicates that the phase averages of stresses and strains are described satisfactorily, this does not necessarily imply that the higher statistical moments of the stress and strain distributions (and thus the heterogeneity of these fields) are captured correctly.

It is important to be aware that work in the field of micromechanics of materials invariably involves finding viable compromises in terms of the complexity of the models, which, on the one hand, have to be able to account (at least approximately) for the physical phenomena relevant to the given problem, and, on the other hand, must be sufficiently simple to allow solutions to be obtained within the relevant constraints of time, cost, and computational resources. Obviously, actual problems cannot be solved without recourse to various approximations and tradeoffs — the important point is to be aware of them and to account for them in interpreting and assessing results. It is worth keeping in mind that there is no such thing as a “best micromechanical approach” to all applications.

\(^{130}\)One important exception are models for studying macroscopic cracks (e.g., via embedded cells), where the crack length does introduce an absolute length scale.

It is worth noting that the behavior of nanocomposites is typically dominated by interfaces (or interphases). Appropriately accounting for this behavior may require the introduction of an absolute length scale. Within the framework of mean-field methods this may be achieved by introducing surface terms into the Eshelby tensor, see, e.g., Duan et al. (2005).
Bibliography


183


184


